From Linear Models to Neural Networks

Linear & logistic regression, neurons, MLPs, universal approximation

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ES 667 · Deep Learning · IIT Gandhinagar

Lecture 1 chose the loss; Lecture 2 expands the prediction rule

Keep from Lecture 1

Regression output $\rightarrow$ Gaussian model $\rightarrow$ MSE

Class probabilities $\rightarrow$ Bernoulli / categorical model $\rightarrow$ cross-entropy

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We will not re-derive the losses; today is about making f_θ more expressive.

Outline

1. **Linear prediction** — weighted sums, batches, and output shapes
2. **One neuron** — add an activation to create a nonlinear feature
3. **The limitation** — see exactly why a linear model cannot solve XOR
4. **An MLP** — compose ReLU features and solve the XOR counterexample
5. **Capacity and depth** — approximate functions, then compose representations

One notation covers every prediction rule

Q QUESTION

Write $f_{\boldsymbol{\theta}}(\boldsymbol{x})$ for the model's prediction at input $\boldsymbol{x}$.

$\boldsymbol{x}$: the input features
$\boldsymbol{\theta}$: all trainable weights and biases
$\boldsymbol{f_{\theta}}$: the rule that maps input to a score or prediction

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Starting point — one weighted sum plus a bias:

$$f_{\boldsymbol{\theta}}(\boldsymbol{x}) = \boldsymbol{w}^{\top}\boldsymbol{x} + b$$

We will build the network gradually; the full nested formula comes only after each part is concrete.

The linear starting point

Start with one weighted sum

For one output, every input node sends a weighted value into one summation node.

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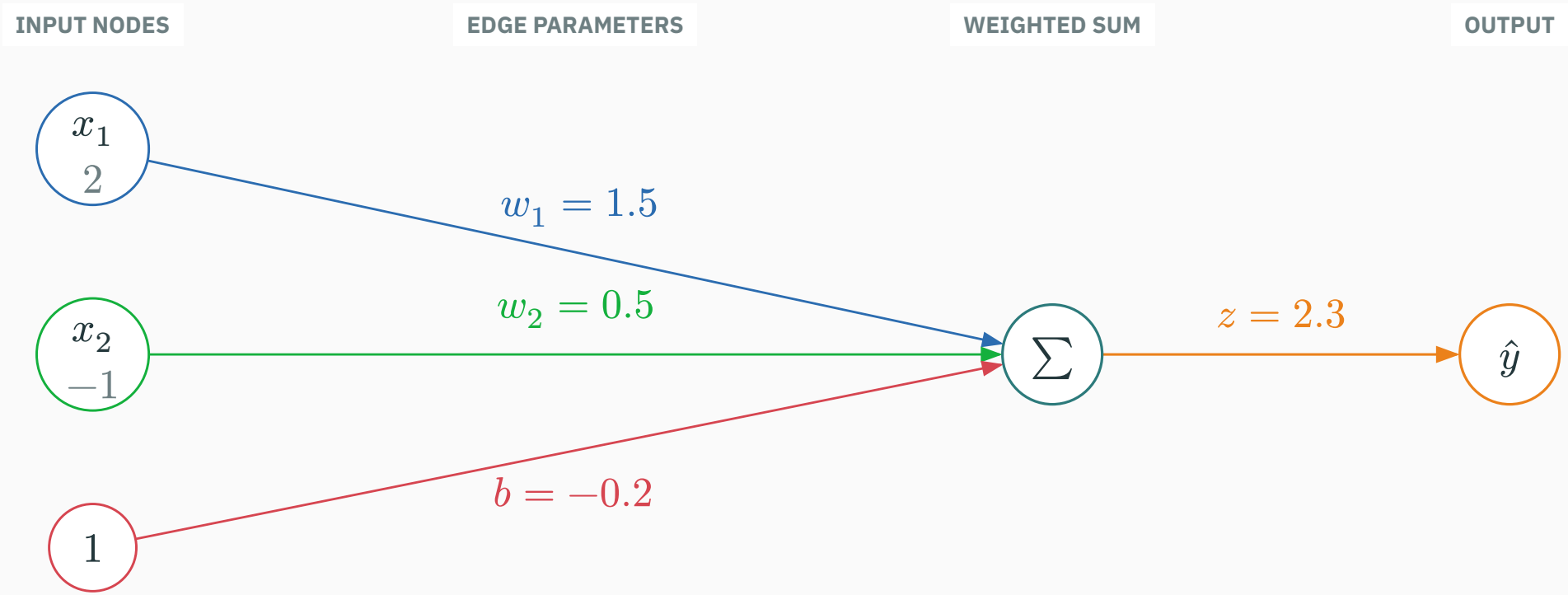

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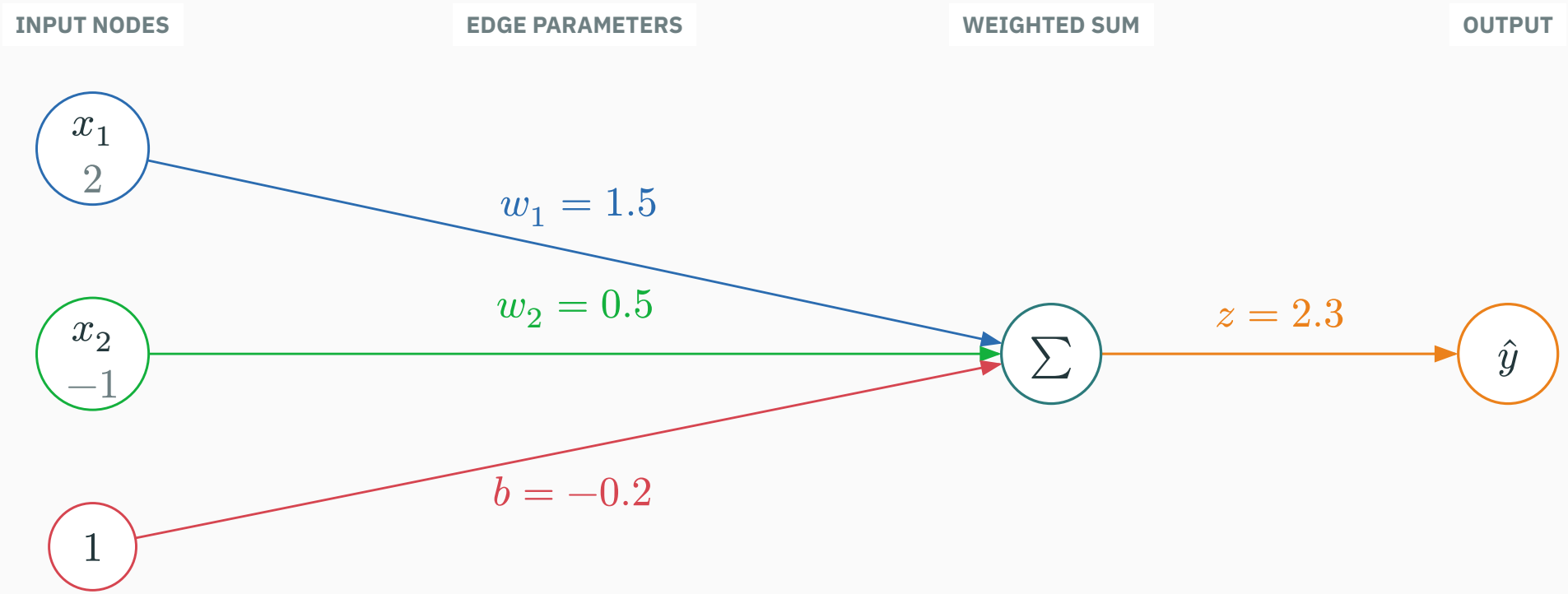
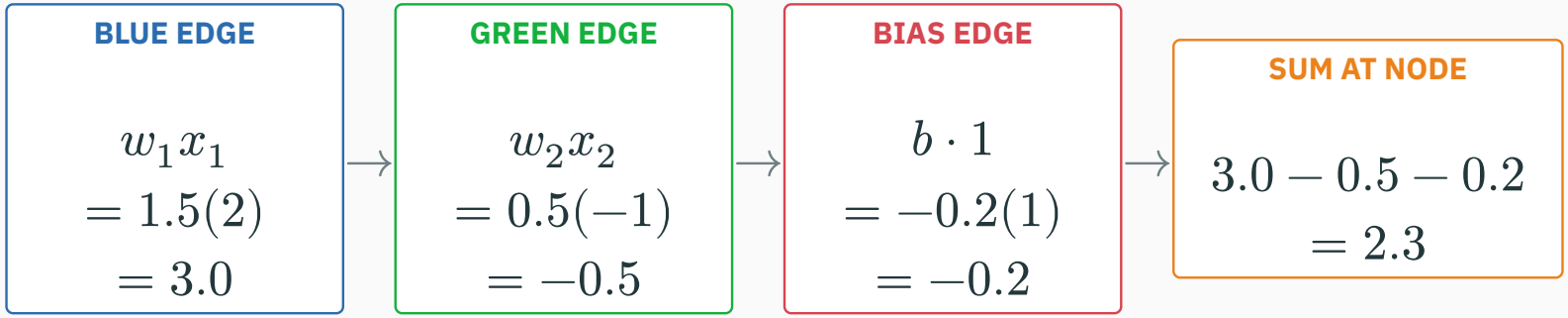


The constant node 1 turns the bias into one more incoming edge: $b \cdot 1$.

Calculate every contribution entering the node

D DERIVATION



Calculate every contribution entering the node

D DERIVATION

BLUE EDGE

$$w_1 x_1 = 1.5(2) = 3.0$$

→

GREEN EDGE

$$w_2 x_2 = 0.5(-1) = -0.5$$

→

BIAS EDGE

$$b \cdot 1 = -0.2(1) = -0.2$$

→

SUM AT NODE

$$3.0 - 0.5 - 0.2 = 2.3$$

$$\hat{y} = \underbrace{1.5(2)}_{3.0} + \underbrace{0.5(-1)}_{-0.5} + \underbrace{(-0.2)(1)}_{-0.2} = 2.3.$$

Calculate every contribution entering the node

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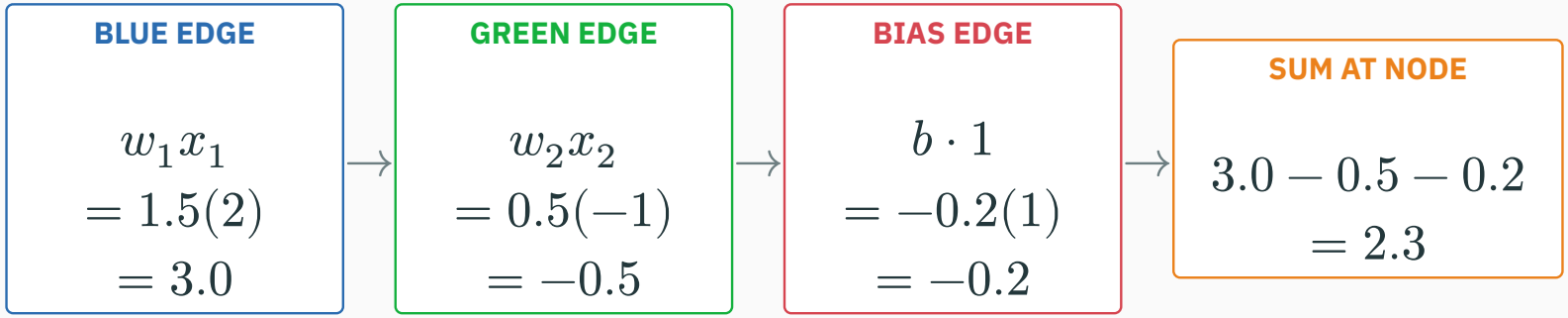

$$\hat{y} = \underbrace{1.5(2)}_{3.0} + \underbrace{0.5(-1)}_{-0.5} + \underbrace{(-0.2)(1)}_{-0.2} = 2.3.$$

Each edge contributes “weight × incoming value”; the node adds all contributions.

A batch is a labelled data matrix

V VISUAL

example	feature x_1	feature x_2	target y
1	2	−1	2.4
2	0	3	1.0
3	−1	1	−1.1

DATA MATRIX X

$$X = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ -1 & 1 \end{pmatrix}$$

3 rows = examples

2 columns = features

TARGET VECTOR Y

$$y = \begin{pmatrix} 2.4 \\ 1.0 \\ -1.1 \end{pmatrix}$$

one observed target per example

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Here $n = 3$ examples and $d = 2$ input features, so $\boldsymbol{X} \in \mathbb{R}^{3\times 2}$ and $\boldsymbol{y} \in \mathbb{R}^3$.

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one observed target per example

Row i contains the input $x^{(i)}$ and its observed answer $y^{(i)}$.

One matrix multiplication predicts all three rows

D DERIVATION

Reuse $\boldsymbol{w} = \begin{pmatrix} 1.5 \\ 0.5 \end{pmatrix}$ and $b = -0.2$ for every example.

One matrix multiplication predicts all three rows

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Reuse $\boldsymbol{w} = \begin{pmatrix} 1.5 \\ 0.5 \end{pmatrix}$ and $b = -0.2$ for every example.

$$\boldsymbol{X}\boldsymbol{w} + b\mathbf{1} = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1.5 \\ 0.5 \end{pmatrix} + (-0.2) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

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ROW 1

$1.5(2) + 0.5(-1) - 0.2 = 2.3$

ROW 2

$1.5(0) + 0.5(3) - 0.2 = 1.3$

ROW 3

$1.5(-1) + 0.5(1) - 0.2 = -1.2$

One matrix multiplication predicts all three rows

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$\hat{y}$ contains predictions; y contains observed targets. They are not the same object.

The bias node becomes a column of ones

V VISUAL

AUGMENT THE DATA MATRIX

$$\tilde{\boldsymbol{X}} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

STACK THE PARAMETERS

$$\boldsymbol{\theta} = \begin{pmatrix} w_1 \\ w_2 \\ b \end{pmatrix} = \begin{pmatrix} 1.5 \\ 0.5 \\ -0.2 \end{pmatrix}$$

$\tilde{\boldsymbol{X}}$

original feature columns
$+$ one constant column

$\boldsymbol{\theta}$

all weights
$+$ the bias

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$$\hat{\boldsymbol{y}} = \tilde{\boldsymbol{X}}\boldsymbol{\theta}, \quad (n \times (d+1))((d+1) \times 1) \to (n \times 1).$$

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all weights
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$$\hat{y} = \tilde{X}\theta, \quad (n \times (d+1))((d+1) \times 1) \to (n \times 1).$$

The column of ones is the batch version of the constant bias node on the previous network diagram.

For regression, one weighted sum gives one flat trend

V VISUAL

$$\hat{y} = \boldsymbol{w}^\top \boldsymbol{x} + b$$

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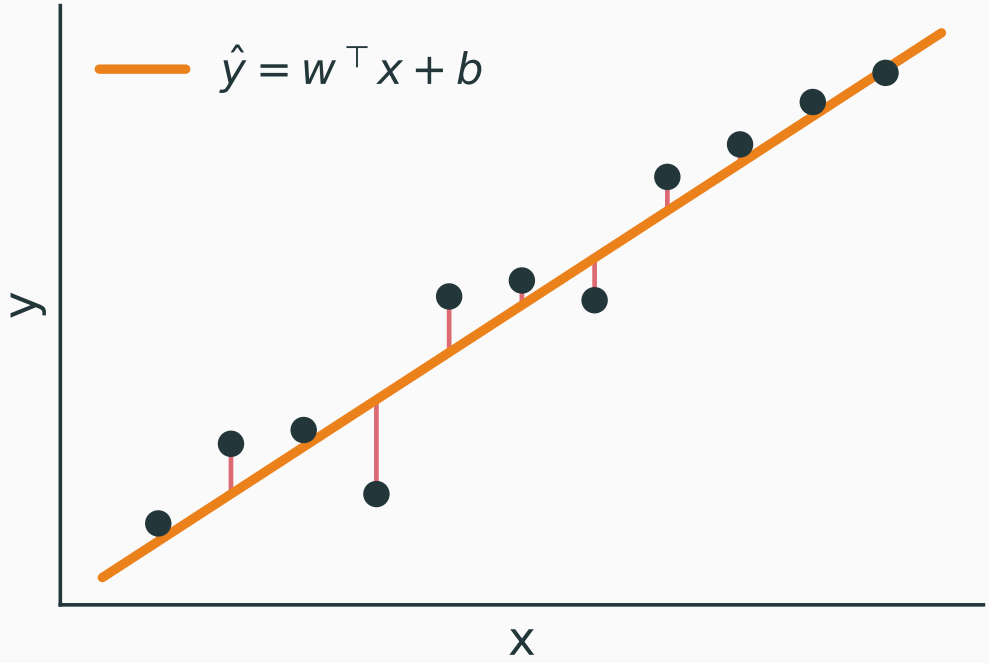

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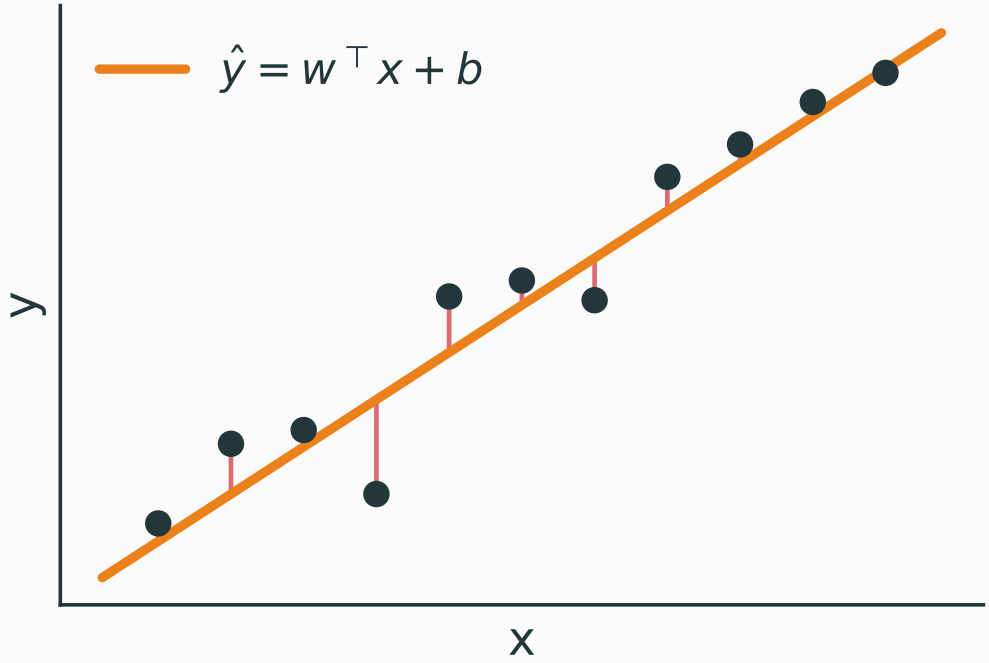

The model can tilt and shift a line or plane, but it cannot bend. Fit it in the Linear GD Colab ↗

From a linear score to a neuron

A neuron has two stages

First compute a weighted sum plus bias; then pass that value through an activation.

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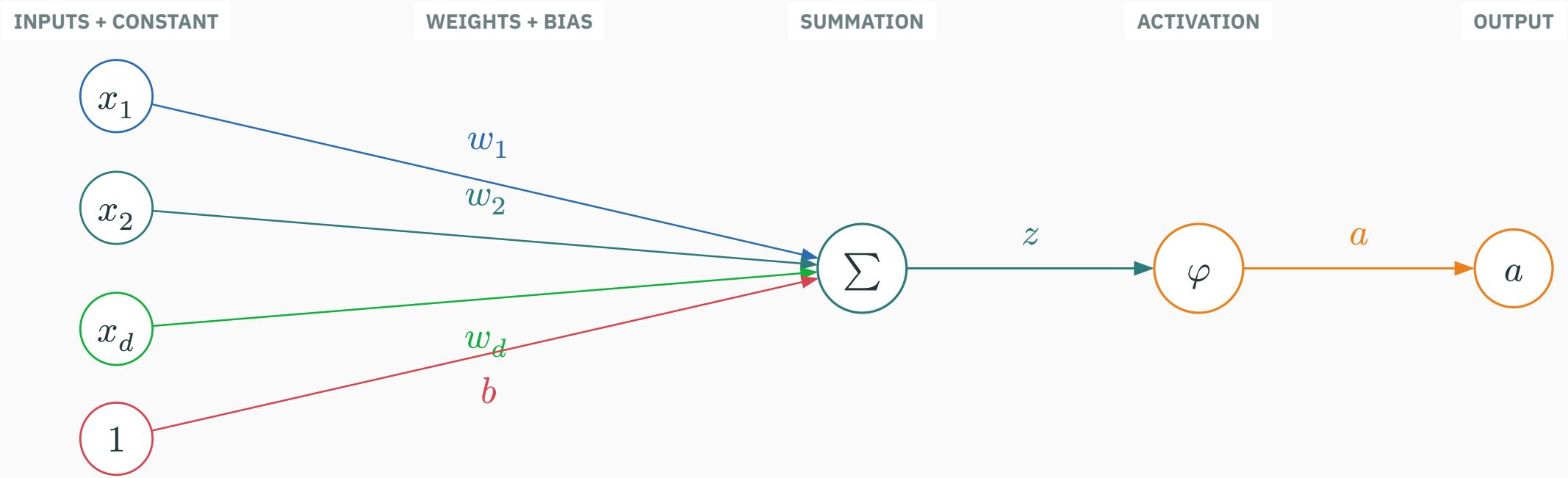

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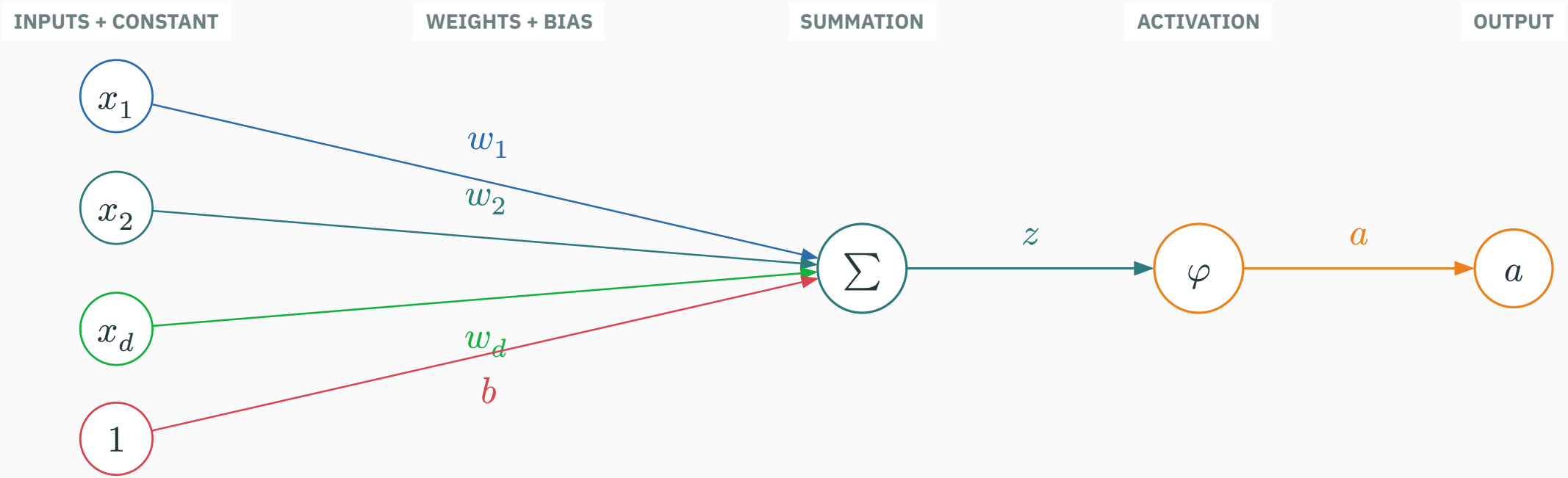

Weights and bias create z at the $\sum$ node; the separate φ node reshapes z into a.

Trace one concrete neuron through the two stages

D DERIVATION

INPUT

$$\boldsymbol{x} = (2, -1)$$

$\rightarrow$

STAGE 1 · SUMMATION

$$\boldsymbol{w} = (1, -1), \quad b = 0$$
$$z = 1(2) + (-1)(-1) + 0 = 3$$

$\rightarrow$

STAGE 2 · ACTIVATION

$$a = \mathrm{ReLU}(3) = 3$$

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Changing $\boldsymbol{w}$ or b changes which input patterns make z large.

Changing φ changes how the same pre-activation z is gated or reshaped.

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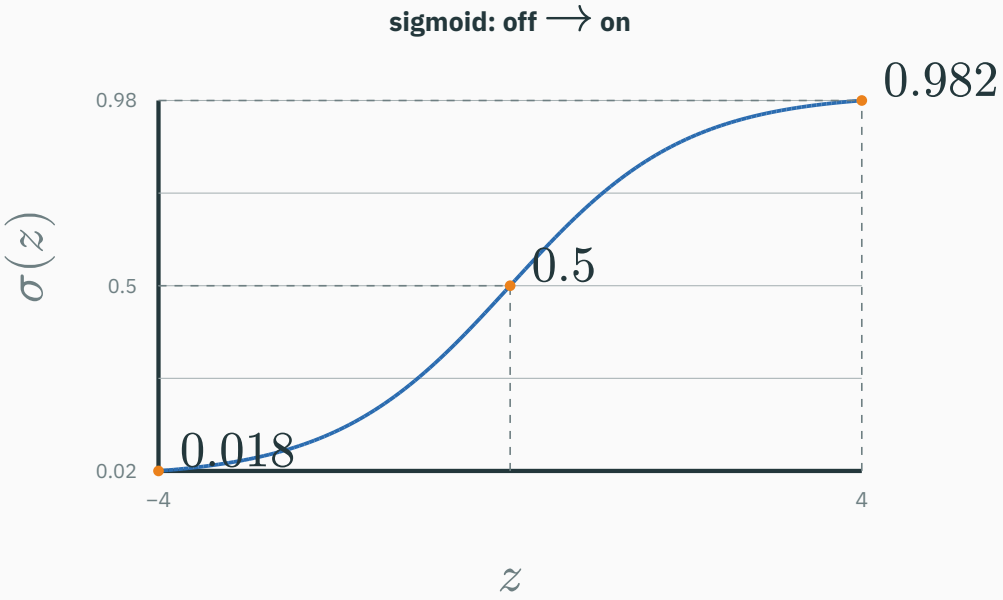
Topology tells us where values flow; this calculation tells us the number carried along each stage.

Bounded activations behave like smooth switches

V VISUAL

$$\sigma(z) = \frac{1}{1+\exp(-z)}$$



Bounded activations behave like smooth switches

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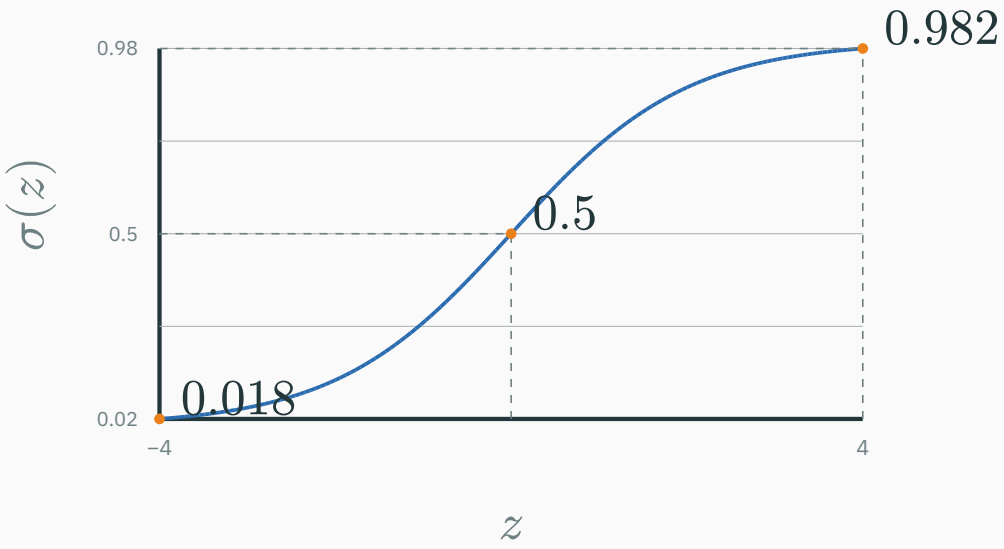
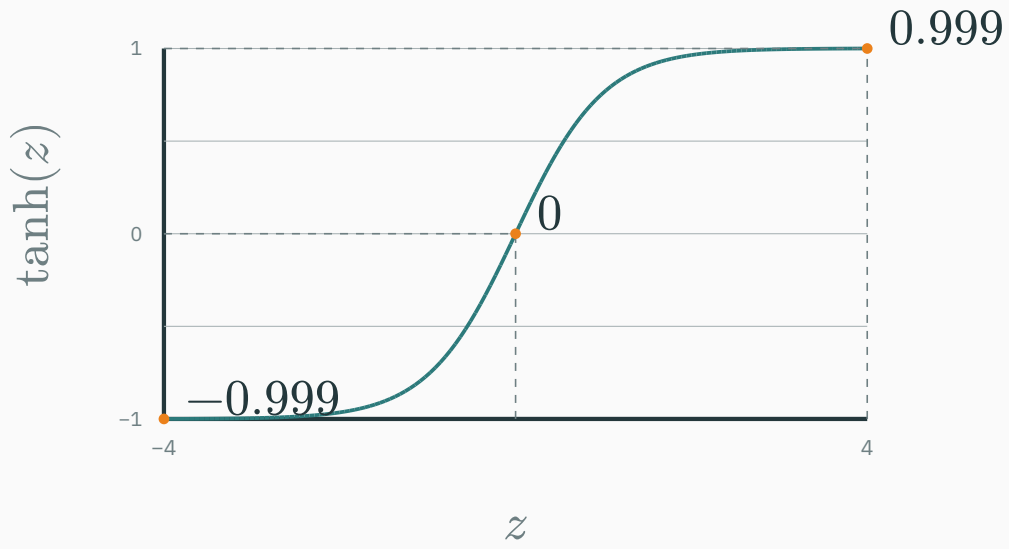
sigmoid: off ⟶ on



$$\tanh(z) = \frac{\exp(z)-\exp(-z)}{\exp(z)+\exp(-z)}$$

tanh: negative ⟶ positive



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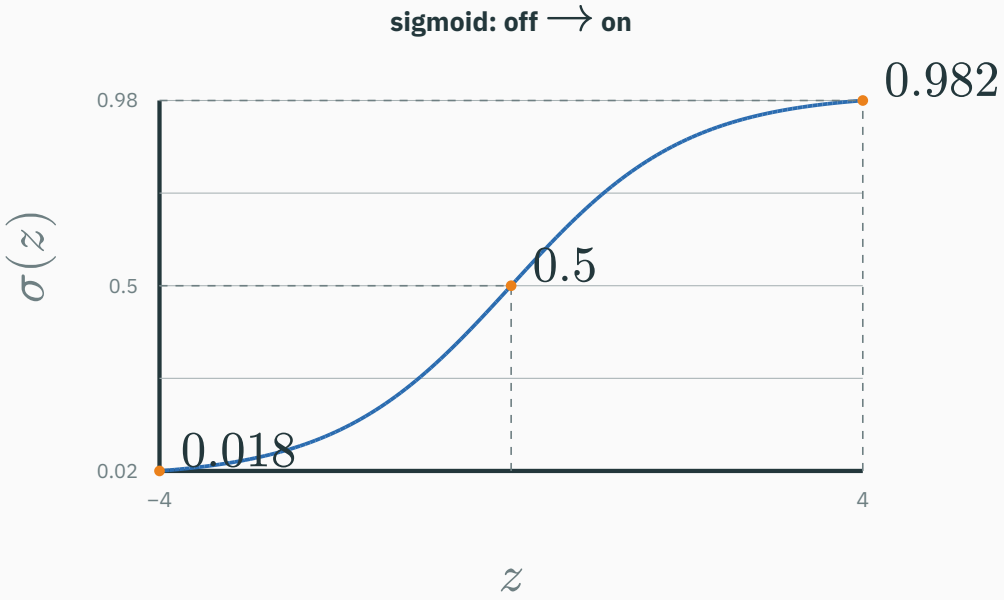
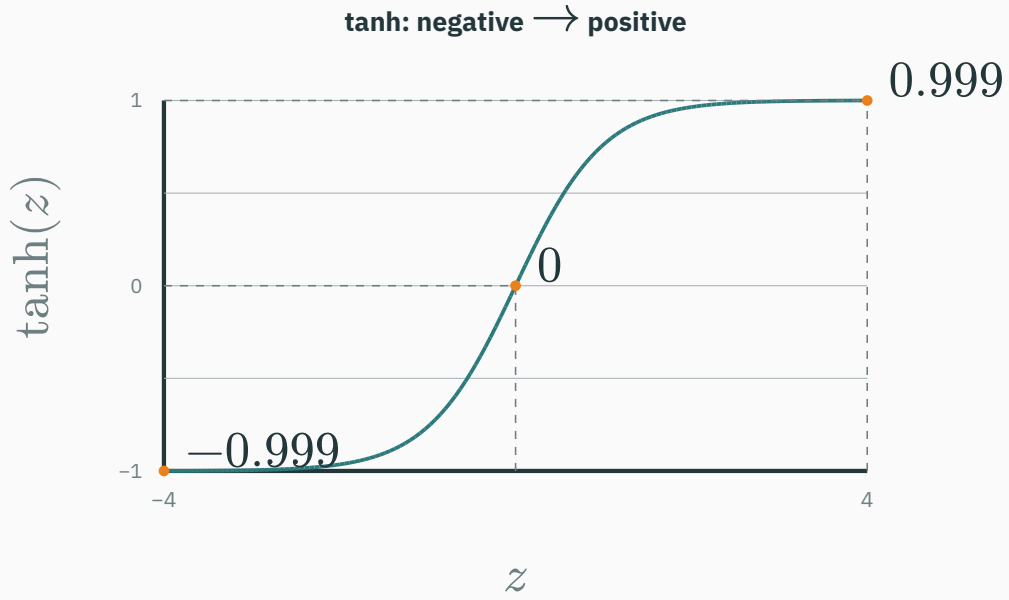


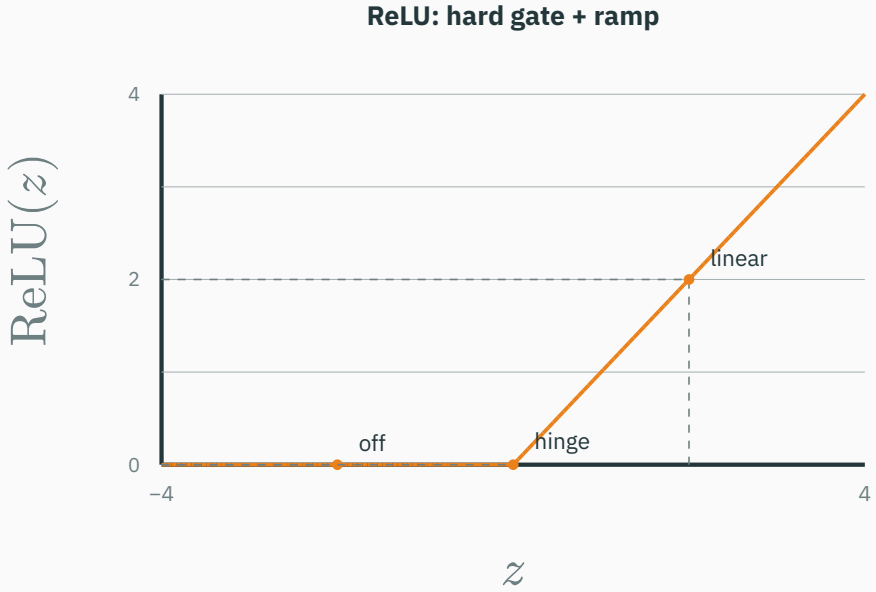


Sigmoid answers “how on?”; tanh answers “negative or positive, and how strongly?”

ReLU and GELU behave like ramps

V VISUAL

$$\mathrm{ReLU}(z) = \max(0, z)$$

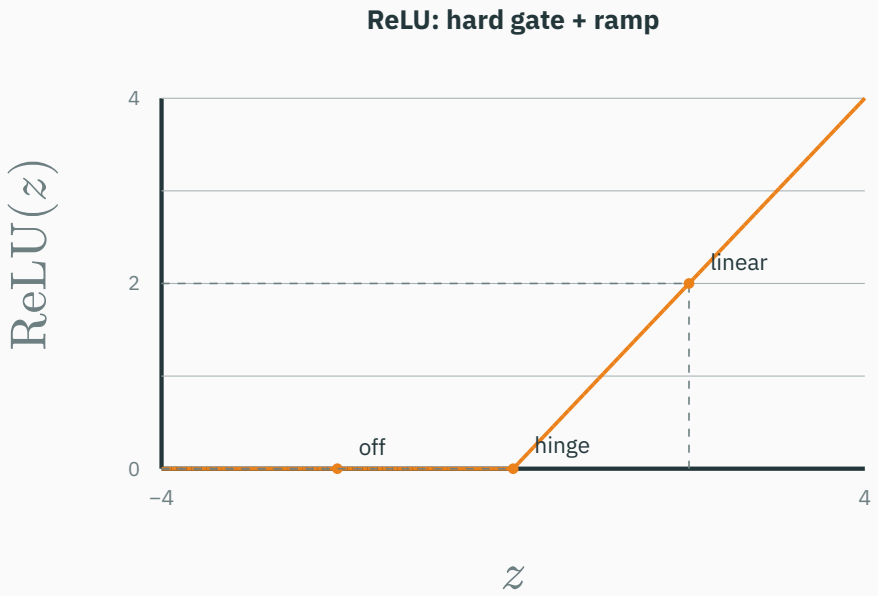


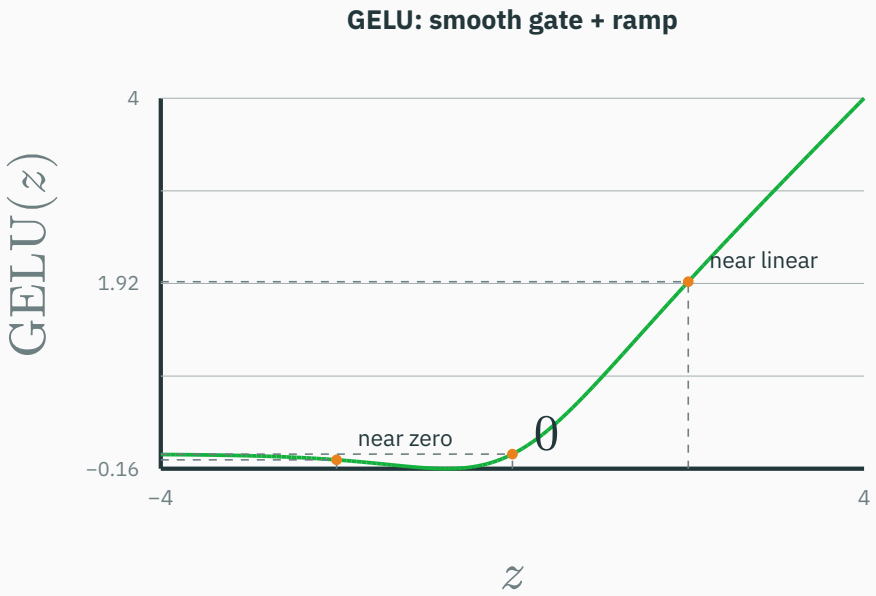
ReLU and GELU behave like ramps

V VISUAL

$$\mathrm{ReLU}(z) = \max(0, z)$$

$$\mathrm{GELU}(z) \approx z\sigma(1.702z)$$





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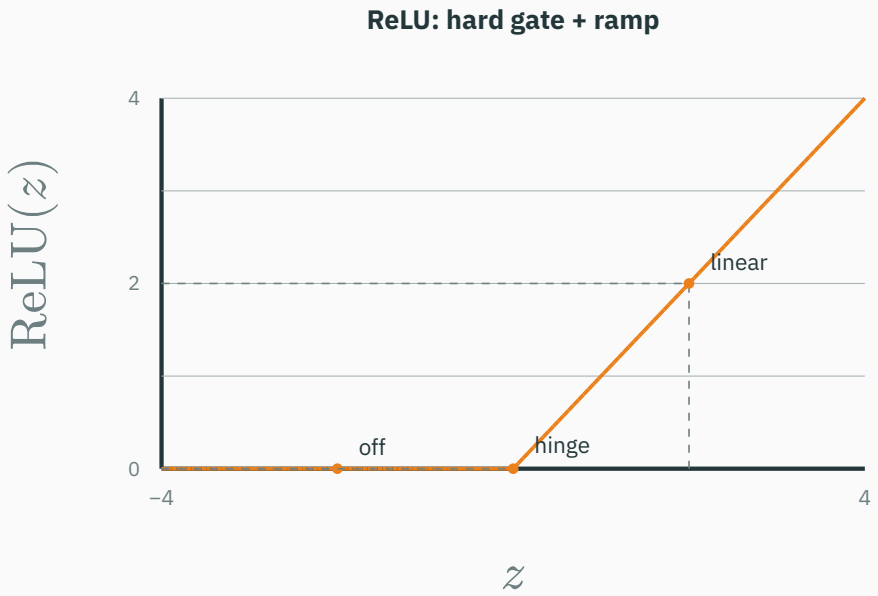
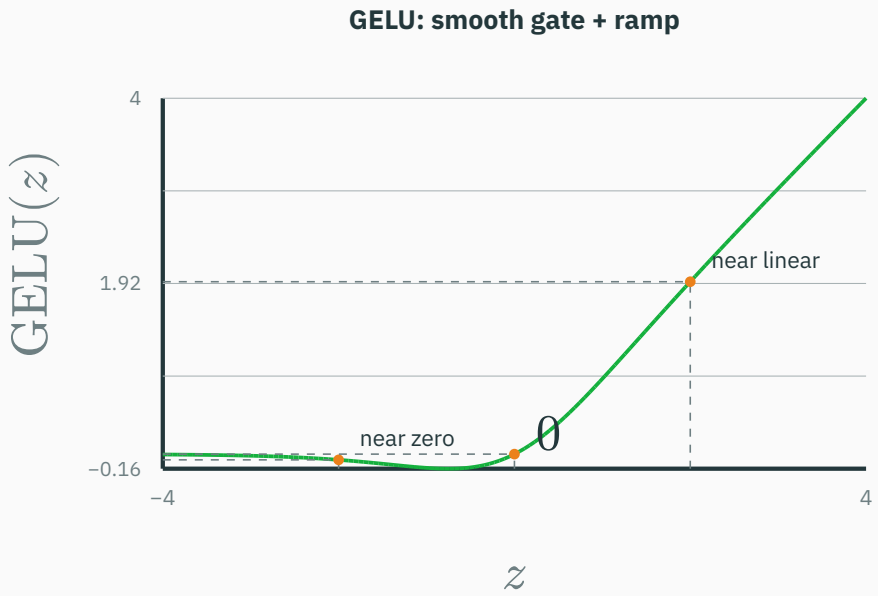
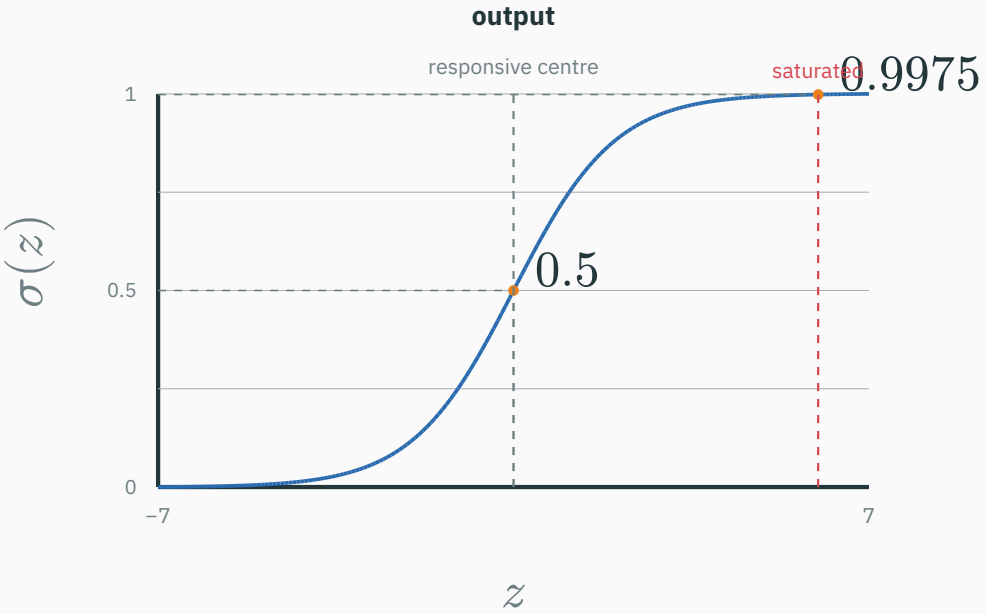




Both build ramp-like hidden features; ReLU has a sharp hinge, while GELU eases through the gate.

Saturation means the curve has become locally flat

V VISUAL



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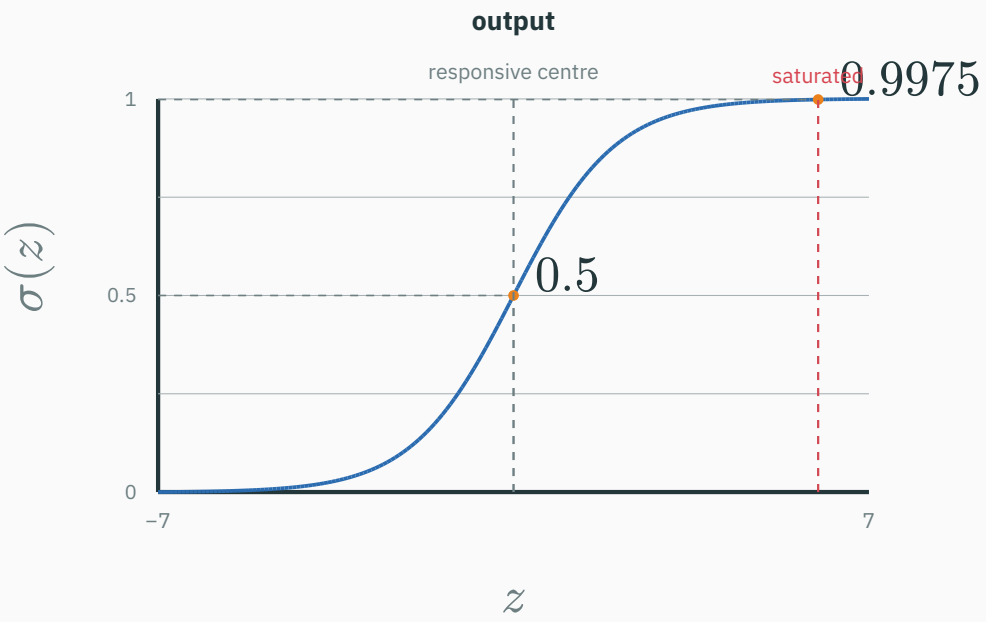

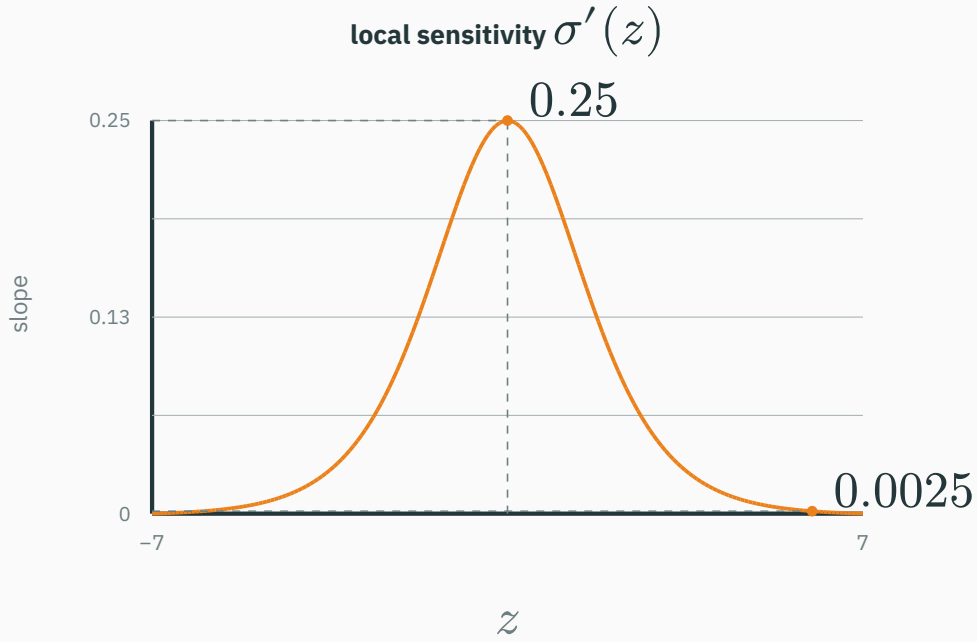

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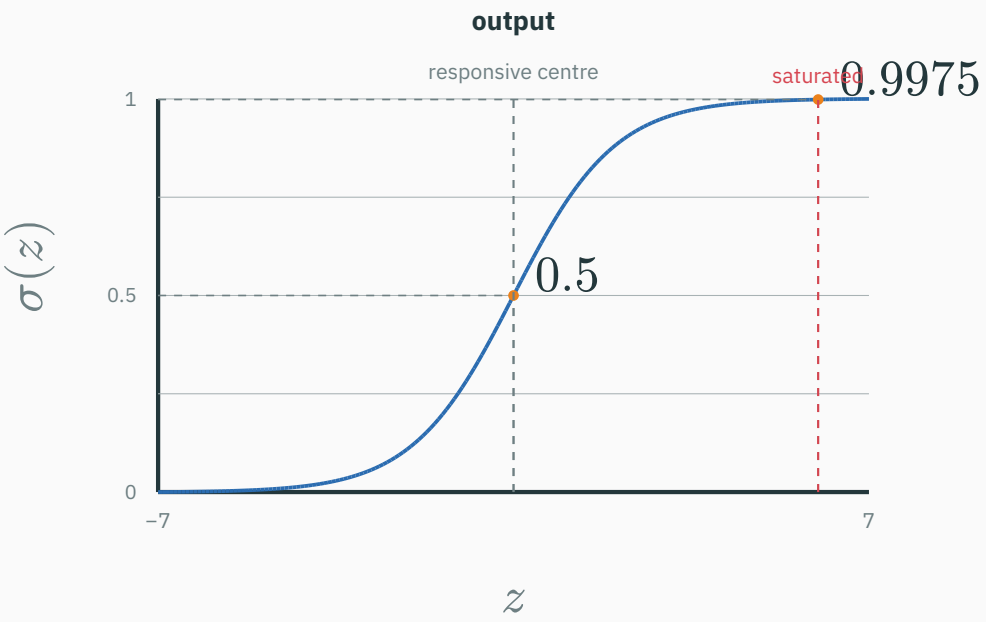

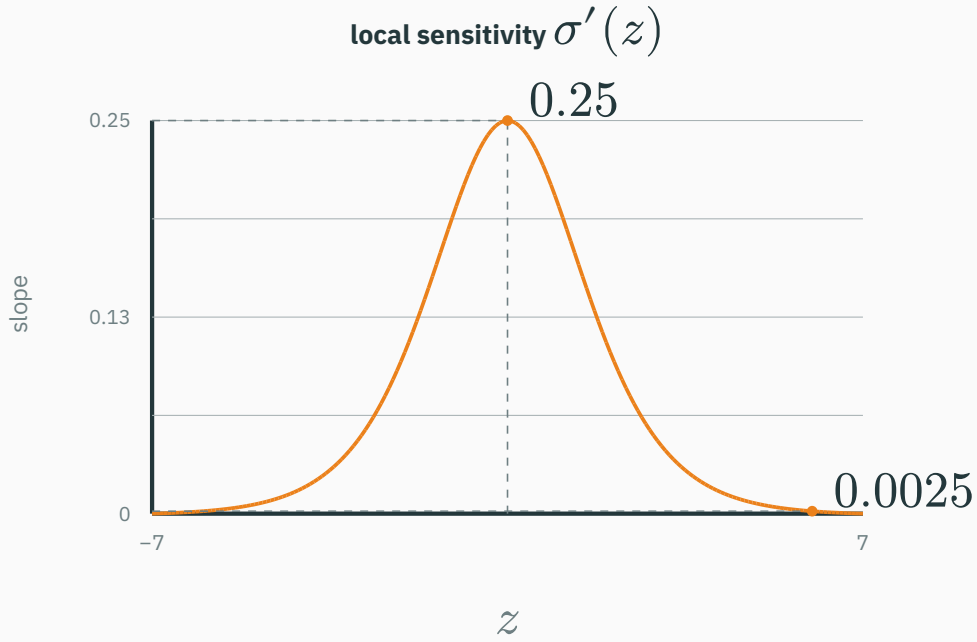

Near $z = 0$, the curve is steep; near $z = 6$, it is almost horizontal.

Saturation means the curve has become locally flat

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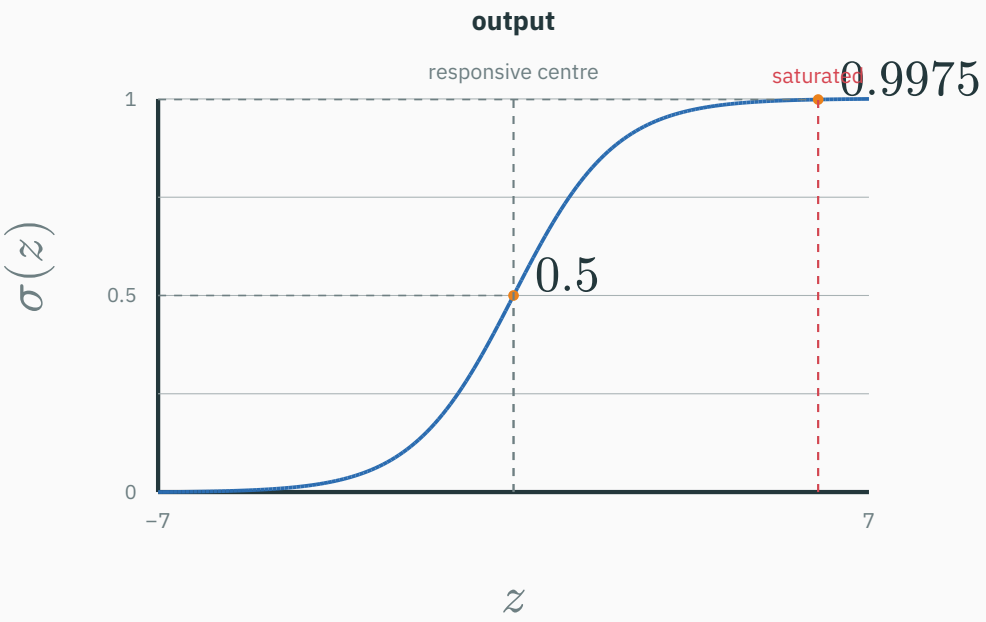

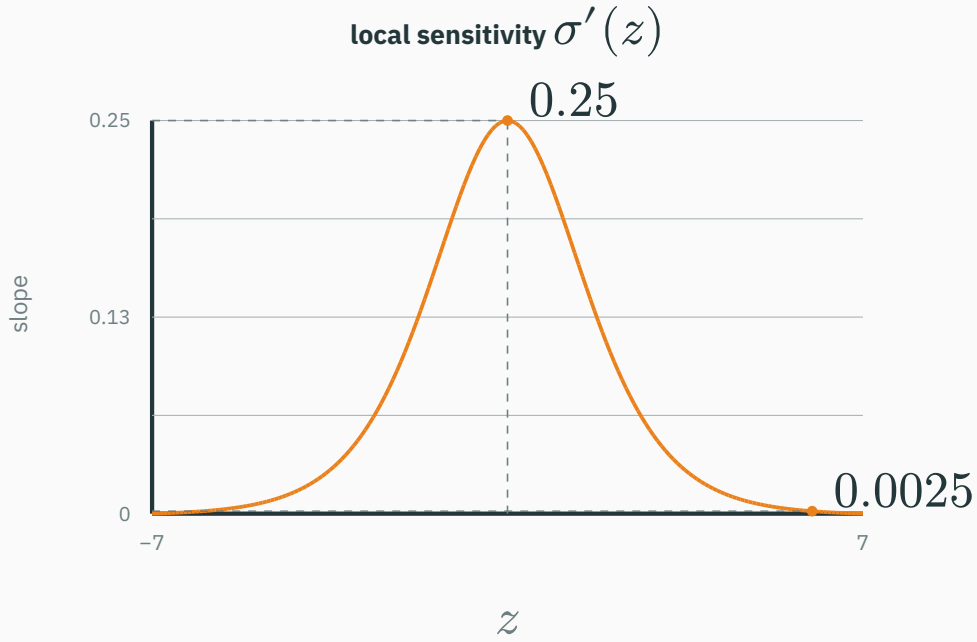

Near $z = 0$, the curve is steep; near $z = 6$, it is almost horizontal.

A saturated unit still outputs a number, but changing its input barely changes that output.

Calculate the effect of saturation

D DERIVATION

CENTRE: Z = 0

$$\sigma'(0) = 0.25$$

for $\Delta z = 0.1$:

$$\Delta h \approx 0.25(0.1) = 0.025$$

same Δz $\rightarrow$

SATURATED: Z = 6

$$\sigma'(6) \approx 0.0025$$

for $\Delta z = 0.1$:

$$\Delta h \approx 0.0025(0.1) = 0.00025$$

Calculate the effect of saturation

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$$\sigma'(6) \approx 0.0025$$

for $\Delta z = 0.1$:

$$\Delta h \approx 0.0025(0.1) = 0.00025$$

$\frac{0.025}{0.00025} = 100 \rightarrow$ **the saturated unit responds 100× less.**

Calculate the effect of saturation

D DERIVATION

CENTRE: Z = 0

$$\sigma'(0) = 0.25$$

for $\Delta z = 0.1$:

$$\Delta h \approx 0.25(0.1) = 0.025$$

same Δz $\rightarrow$

SATURATED: Z = 6

$$\sigma'(6) \approx 0.0025$$

for $\Delta z = 0.1$:

$$\Delta h \approx 0.0025(0.1) = 0.00025$$

$\frac{0.025}{0.00025} = 100 \rightarrow$ **the saturated unit responds 100× less.**

For a positive ReLU input, $\mathrm{ReLU}'(z) = 1$, so the same $\Delta z = 0.1$ gives $\Delta h \approx 0.1$.

Calculate the effect of saturation

D DERIVATION

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For a positive ReLU input, $\mathrm{ReLU}'(z) = 1$, so the same $\Delta z = 0.1$ gives $\Delta h \approx 0.1$.

The derivative is a local responsiveness meter; Lecture 3 will connect it to the learning signal.

Question: do more linear layers create new shapes?

Q QUESTION

Remove every activation. What can a stack of “weighted sum + bias” layers represent?

A. One equivalent weighted sum + bias

B. Any curved decision boundary

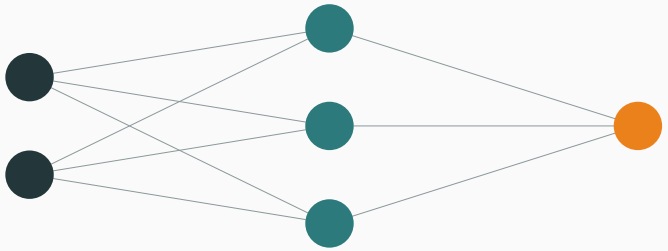
C. Only a constant function

D. A probability distribution automatically

Answer in nodes: two linear layers collapse into one

A ANSWER

TWO LINEAR LAYERS

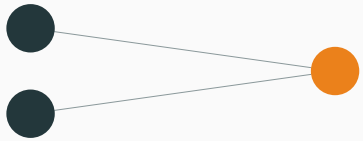


input x linear h output y

$$h = W_1 x + b_1$$
$$y = W_2 h + b_2$$

≡

ONE EQUIVALENT LAYER



input x output y

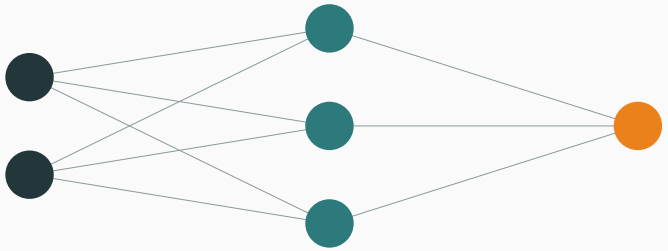
$$y = W_{\text{eq}} x + b_{\text{eq}}$$

no hidden representation is needed

Answer in nodes: two linear layers collapse into one

A ANSWER

TWO LINEAR LAYERS



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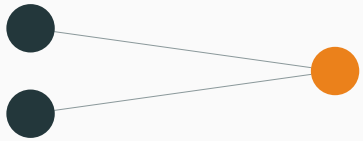

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$$W_{\text{eq}} = W_2 W_1, \quad b_{\text{eq}} = W_2 b_1 + b_2$$

Answer in nodes: two linear layers collapse into one

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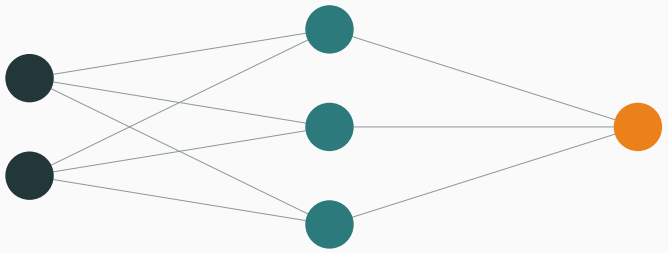
TWO LINEAR LAYERS



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no hidden representation is needed

$$W_{\text{eq}} = W_2 W_1, \quad b_{\text{eq}} = W_2 b_1 + b_2$$

Without a φ node, the middle column adds parameters but no new bend or feature shape.

Interactive: activation functions

I INTERACTIVE

INTERACTIVE

↗ nipunbatra.github.io/dl-teaching/interactives/activation-explorer.html

Compare activations and their **derivatives**, and see where gradients vanish.
Controls: activation type · input z · show derivative · compare saturation.

Interactive: activation functions

I INTERACTIVE

INTERACTIVE

↗ nipunbatra.github.io/dl-teaching/interactives/activation-explorer.html

Compare activations and their **derivatives**, and see where gradients vanish.
Controls: activation type · input z · show derivative · compare saturation.

Explore large positive and negative logits, where sigmoid's derivative becomes tiny.

Where a single weighted sum fails

A binary classifier draws one straight boundary

V VISUAL

LINEAR SCORE

$$z = \boldsymbol{w}^\top \boldsymbol{x} + b$$

THRESHOLD

$$\hat{y} = 1 \text{ if } z \geq 0$$
$$\hat{y} = 0 \text{ if } z < 0$$

The boundary is where $z = 0$.

A binary classifier draws one straight boundary

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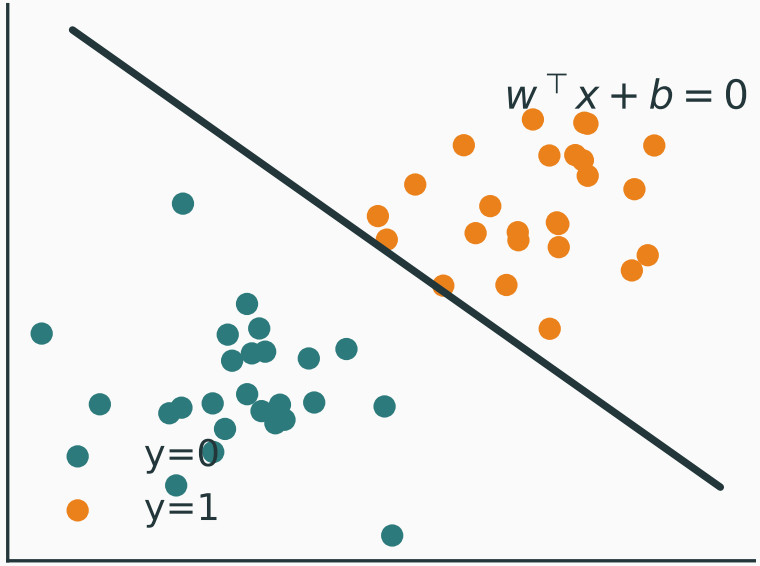

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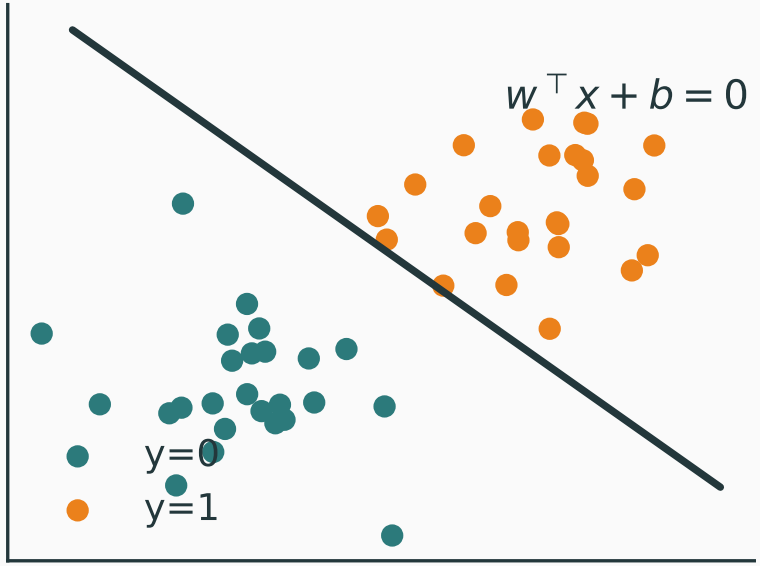

$w^\top x + b = 0$ is one line in 2-D, so binary linear classification works only when one line can separate the classes.

XOR is the smallest pattern that defeats one line

V VISUAL

CLASS A

$(0, 0)$ and $(1, 1)$

CLASS B

$(1, 0)$ and $(0, 1)$

Every proposed line leaves one same-class corner on the wrong side.

XOR is the smallest pattern that defeats one line

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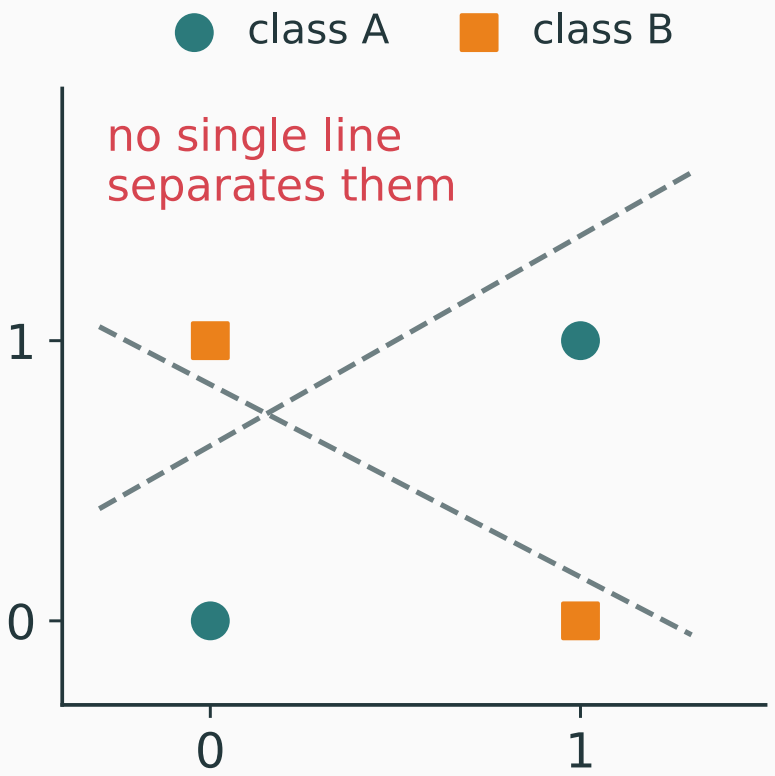

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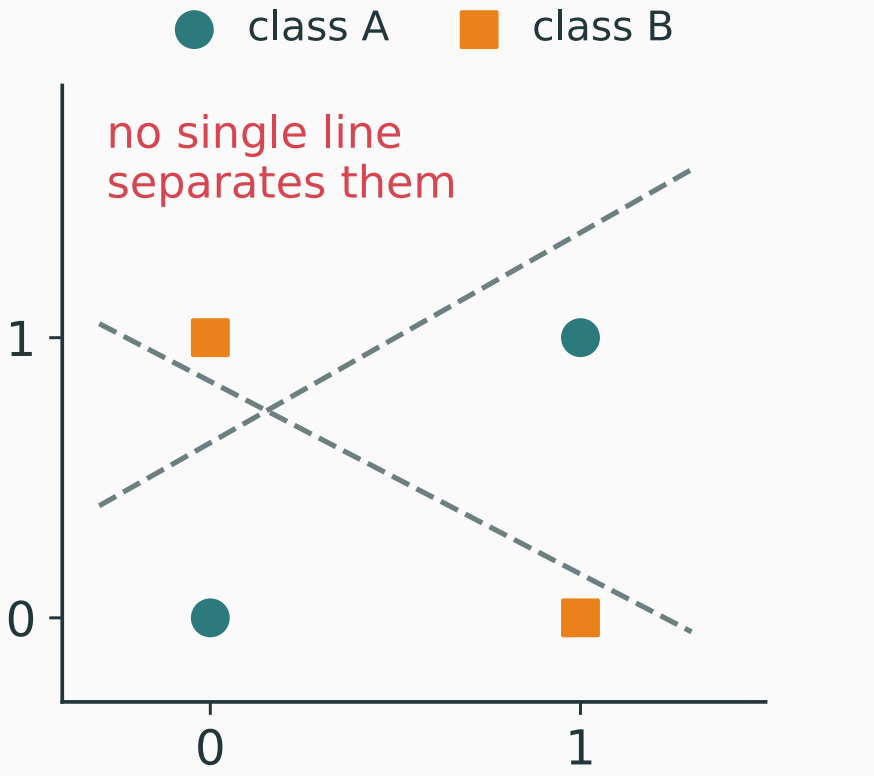

XOR requires a nonlinear feature transformation before a final straight separator can work.

Return to multi-class: several scores create straight-edged regions

V VISUAL

For each class k, compute one linear score $z_{k(\boldsymbol{x})} = \boldsymbol{w}_k^\top \boldsymbol{x} + b_k$.

Return to multi-class: several scores create straight-edged regions

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For each class k, compute one linear score $z_{k(\boldsymbol{x})} = \boldsymbol{w}_k^\top \boldsymbol{x} + b_k$. Predict the largest score: $\hat{y}(\boldsymbol{x}) = \arg\max_k z_{k(\boldsymbol{x})}$.

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Each pair of classes can exchange the lead only where their two scores tie.

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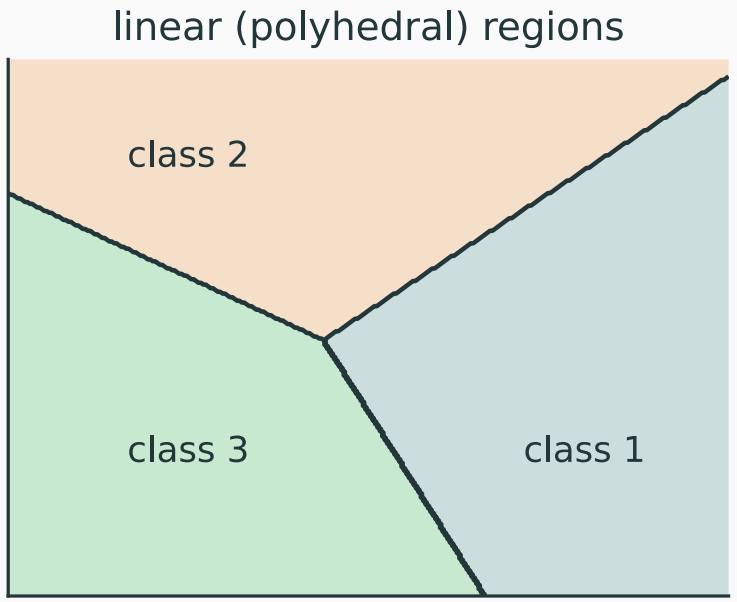

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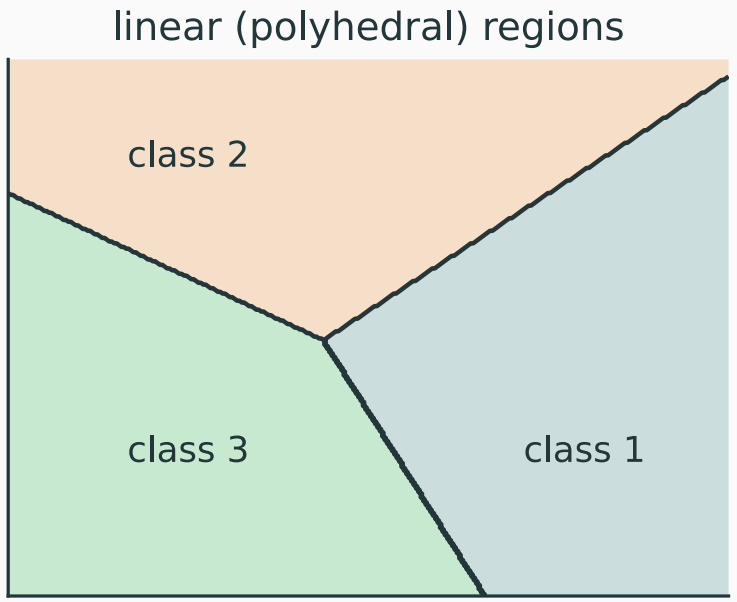
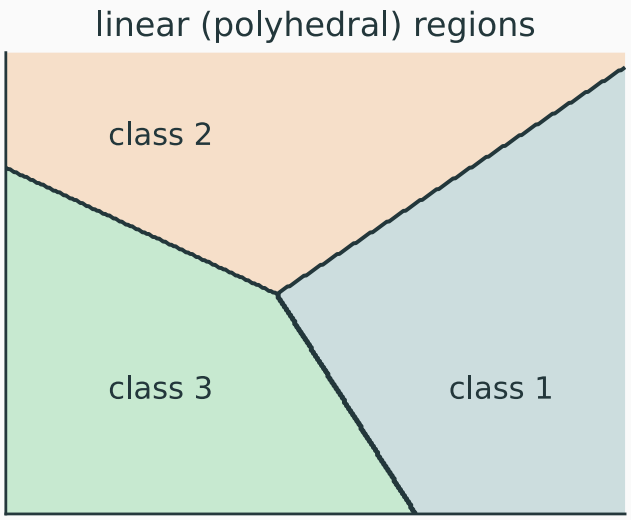


Multi-class regions can form polygons, but every edge still comes from equality between two linear scores.

Question: why are the multi-class boundaries straight?

Q QUESTION



three linear scores divide the input plane

The classifier predicts

$$\hat{y}(\boldsymbol{x}) = \arg\max_k p_{k(\boldsymbol{x})}.$$

When is a point on the boundary between predicted classes i and j?

A. Whenever $p_i = p_j$
B. When $p_i = p_j$ and both are maximal
C. Whenever $p_i + p_j = 1$
D. Only when all K probabilities are equal

Softmax preserves the winner

A ANSWER

SCORES

$$z_i = 2, \quad z_j = 1$$

$\rightarrow$

SOFTMAX PROBABILITIES

$$p_i \approx 0.731, \quad p_j \approx 0.269$$

Softmax preserves the winner

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$$\frac{p_i}{p_j} = \frac{\exp(z_i)}{\exp(z_j)} = \exp\big(z_i - z_j\big)$$

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EQUAL SCORES

$$z_i = z_j \leftrightarrow p_i = p_j$$

$\rightarrow$

LARGER SCORE

$$z_i > z_j \leftrightarrow p_i > p_j$$

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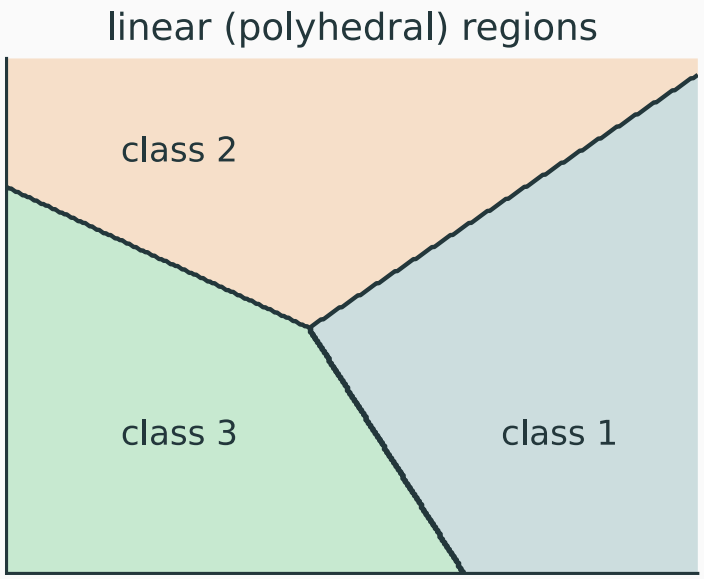
Softmax changes score scale, not score order: $\arg\max_k p_k = \arg\max_k z_k$.

Linear scores can tie only on a flat set

A ANSWER



Each coloured region is where one linear score is largest.

For class k, $z_{k(\boldsymbol{x})} = \boldsymbol{w}_k^\top \boldsymbol{x} + b_k$.

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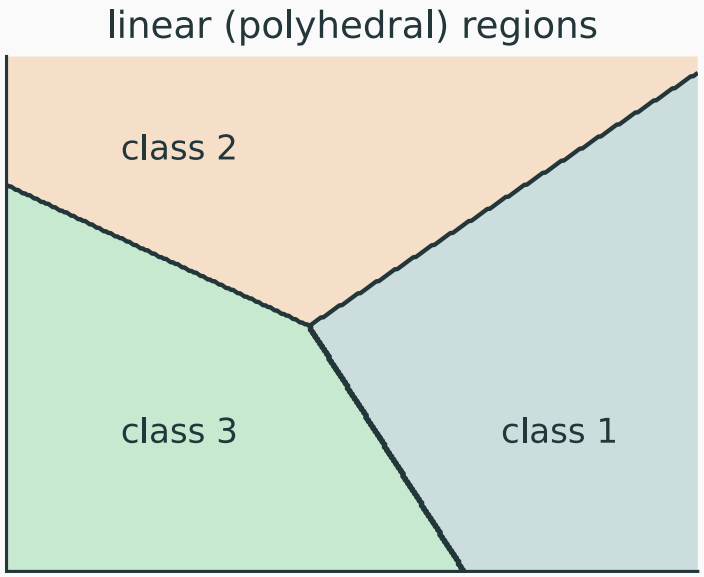
A ANSWER



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For class k, $z_{k(\boldsymbol{x})} = \boldsymbol{w}_k^\top \boldsymbol{x} + b_k$. Two classes tie when $z_{i(\boldsymbol{x})} = z_{j(\boldsymbol{x})}$, so $(\boldsymbol{w}_i - \boldsymbol{w}_j)^\top \boldsymbol{x} + (b_i - b_j) = 0$.

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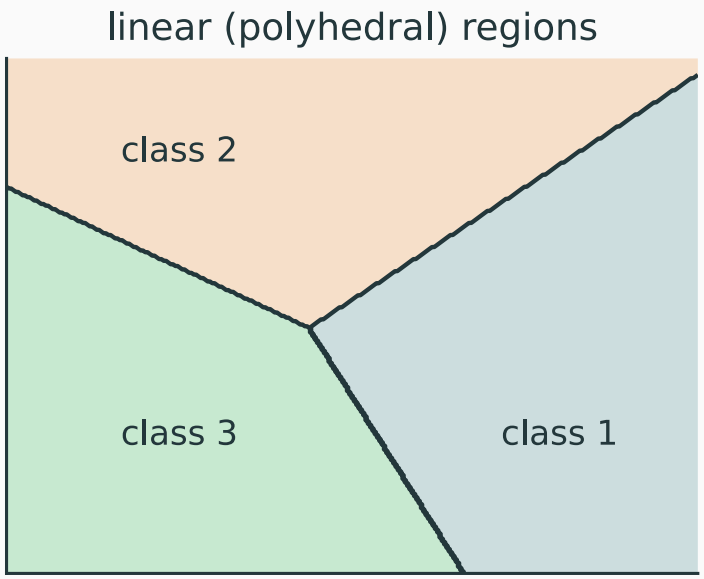

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This is a line in 2-D, a plane in 3-D, and a hyperplane in d dimensions.

Linear scores can tie only on a flat set

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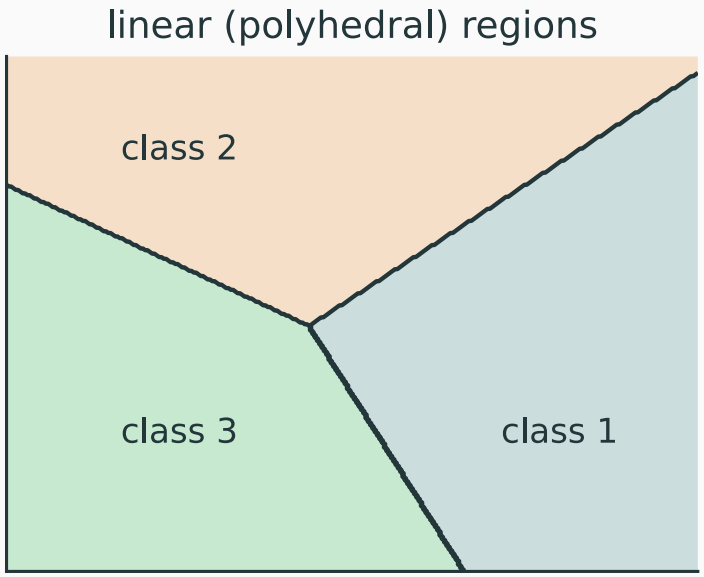

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This is a line in 2-D, a plane in 3-D, and a hyperplane in d dimensions.

Equal linear scores create a candidate straight boundary; softmax cannot bend it.

A concrete example: three linear scores compete

V VISUAL

CLASS 1 SCORE

$$z_1(x) = x$$

CLASS 2 SCORE

$$z_2(x) = -x$$

CLASS 3 SCORE

$$z_3(x) = 0.5$$

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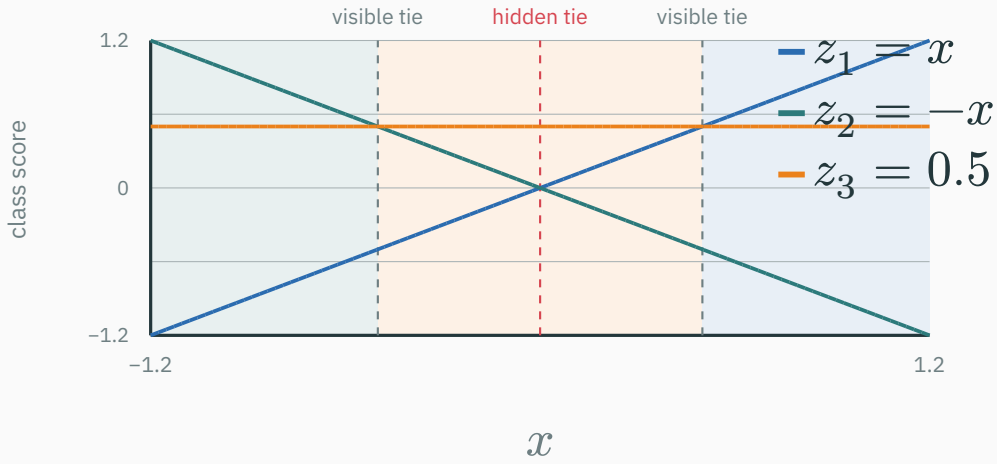

PREDICTED CLASS $= \arg\max_k z_{k(x)}$ **· boundaries at** $x = \pm 0.5$

2: $x < -.5$ 3: $|x| < .5$ 1: $x > .5$

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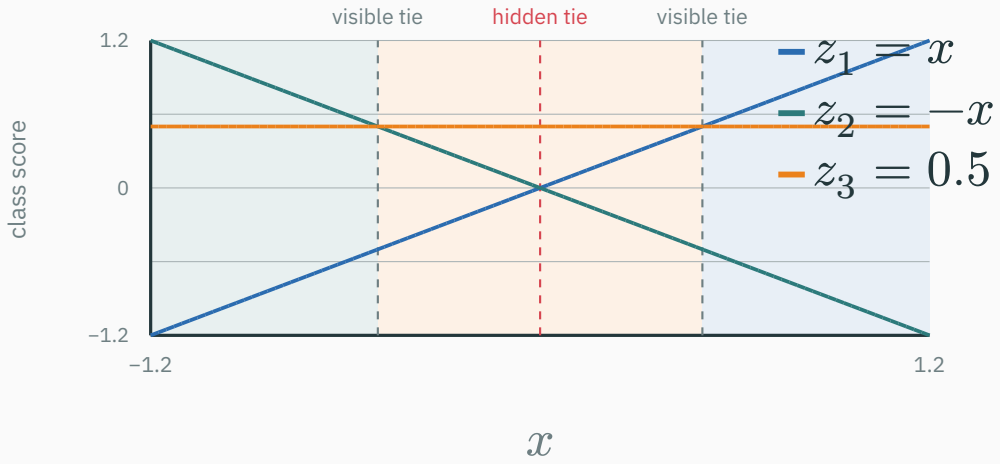

PREDICTED CLASS $= \arg\max_k z_{k(x)}$ **· boundaries at** $x = \pm 0.5$

2: $x < -.5$ 3: $|x| < .5$ 1: $x > .5$

The learned boundaries are $x = -0.5$ and $x = 0.5$; the class-1/class-2 tie at $x = 0$ remains hidden.

Evaluate five explicit inputs to see which ties are visible

D DERIVATION

Prediction $= \arg\max_k z_{k(x)}$; a boundary appears only when the tied scores are maximal.

x	$z_1 = x$	$z_2 = -x$	$z_3 = 0.5$	what is visible?
-1	-1	1	0.5	class 2 wins
-0.5	-0.5	0.5	0.5	**visible boundary 2/3**
0	0	0	0.5	**class 3 wins; 1/2 tie hidden**
0.5	0.5	-0.5	0.5	**visible boundary 1/3**
1	1	-1	0.5	class 1 wins

Evaluate five explicit inputs to see which ties are visible

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0.5	0.5	-0.5	0.5	visible boundary 1/3
1	1	-1	0.5	class 1 wins

The candidate 1/2 tie occurs at $x = 0$, but the visible boundaries occur at $x = -0.5$ and $x = 0.5$.

Candidate boundary versus visible boundary

A ANSWER

CANDIDATE PAIRWISE TIE

$$\mathcal{T}_{ij} = \{\boldsymbol{x} : z_i = z_j\}$$

an entire hyperplane

keep only winners →

VISIBLE CLASS BOUNDARY

$$\mathcal{B}_{ij} = \{\boldsymbol{x} : z_i = z_j \geq z_l \quad \text{for every} \quad l\}$$

the top-scoring portion

Candidate boundary versus visible boundary

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the top-scoring portion

In probability language: $p_i = p_j$ **and both are maximal**.

Candidate boundary versus visible boundary

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the top-scoring portion

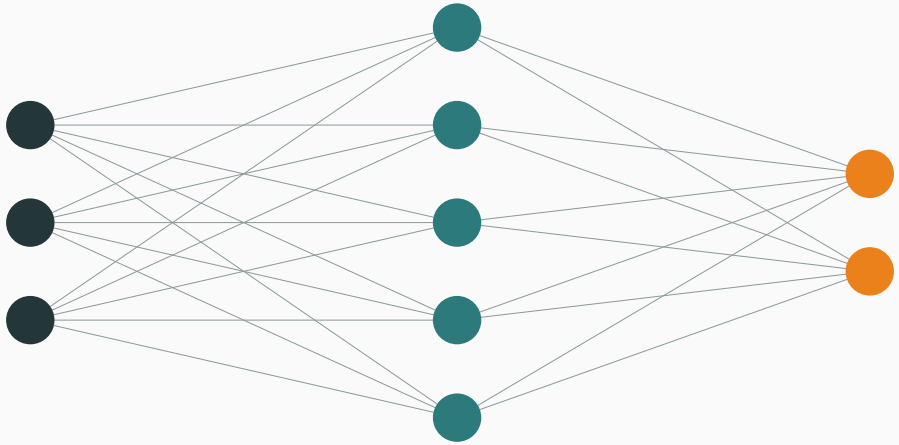
In probability language: $p_i = p_j$ **and both are maximal**.

Correct: B. Multi-class linear classifiers have piecewise-straight visible boundaries.

The multilayer perceptron

A hidden layer is a new column of learned features

V VISUAL



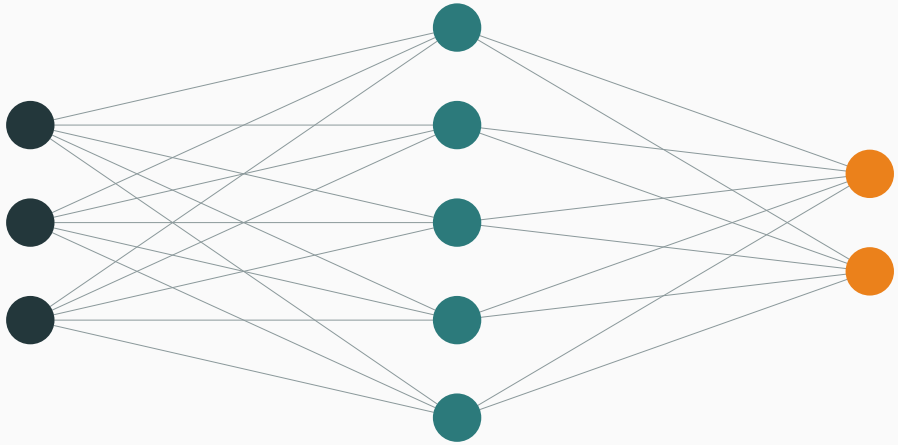
3 input features

5 learned features

2 class scores

A hidden layer is a new column of learned features

V VISUAL



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$$\boldsymbol{h} = \varphi(\boldsymbol{W}_1\boldsymbol{x} + \boldsymbol{b}_1), \quad \boldsymbol{z} = \boldsymbol{W}_2\boldsymbol{h} + \boldsymbol{b}_2.$$

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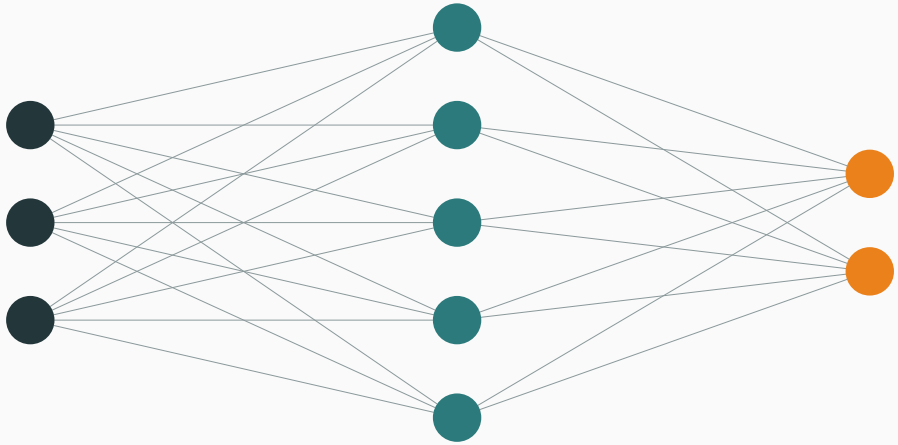
V VISUAL



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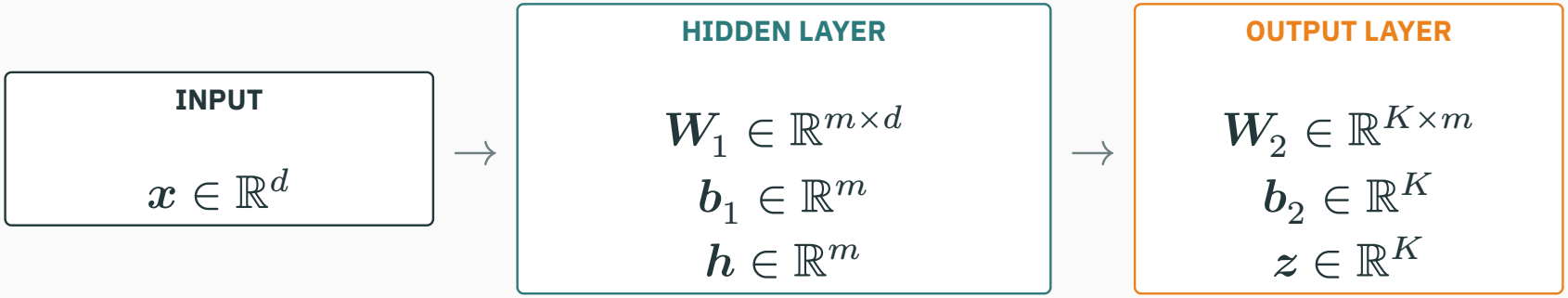
$$\boldsymbol{h} = \varphi(\boldsymbol{W}_1 \boldsymbol{x} + \boldsymbol{b}_1), \quad \boldsymbol{z} = \boldsymbol{W}_2 \boldsymbol{h} + \boldsymbol{b}_2.$$

$\boldsymbol{W}_1$ stores the $3 \times 5 = 15$ blue-to-teal edges; $\boldsymbol{W}_2$ stores the $5 \times 2 = 10$ teal-to-orange edges.

General layer dimensions must agree

D DERIVATION



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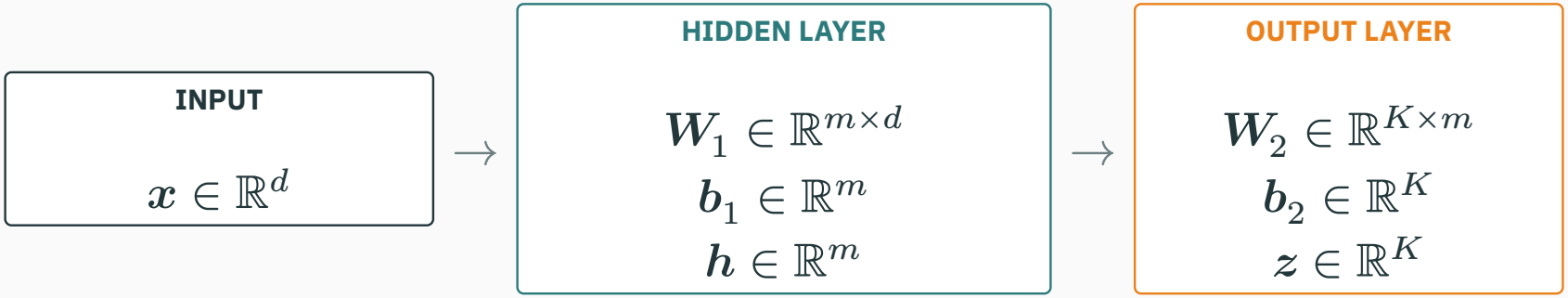
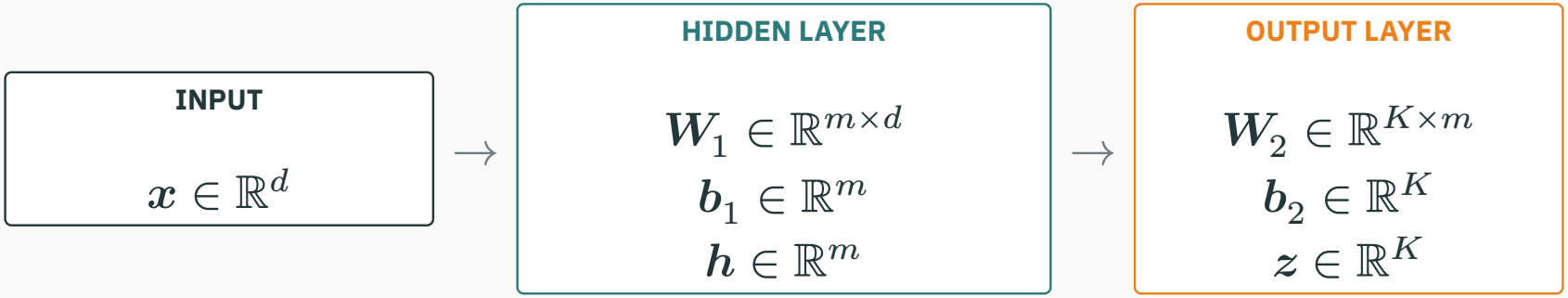


$$(m \times d)(d \times 1) \to (m \times 1), \quad (K \times m)(m \times 1) \to (K \times 1)$$

General layer dimensions must agree

D DERIVATION



$$(m \times d)(d \times 1) \to (m \times 1), \quad (K \times m)(m \times 1) \to (K \times 1)$$

The inner dimensions cancel; trainable scalars $= m(d+1) + K(m+1)$**.**

One XOR network: keep these two features throughout

V VISUAL

Both hidden units receive the same pair $\boldsymbol{x} = (x_1, x_2)$:

$$h_1 = \mathrm{ReLU}(x_1 + x_2), \quad h_2 = \mathrm{ReLU}(x_1 + x_2 - 1)$$

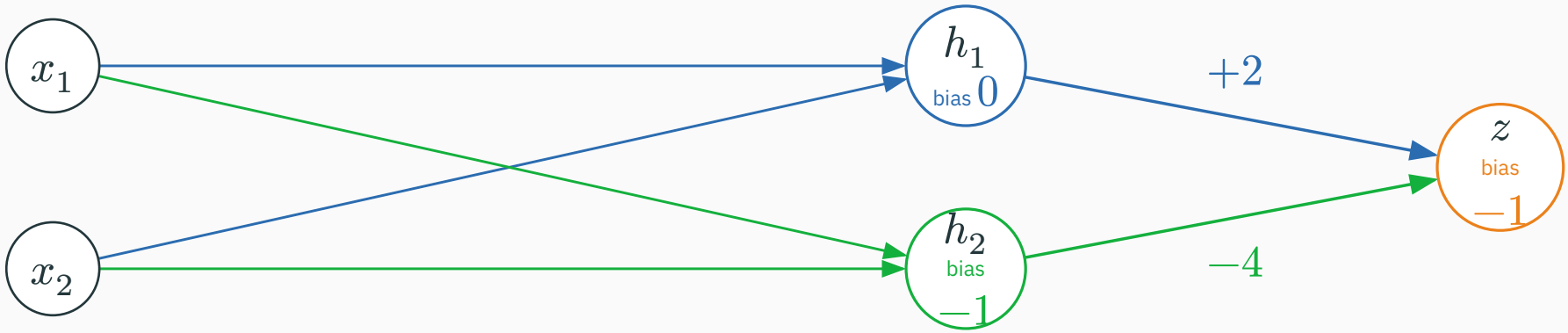
One XOR network: keep these two features throughout

V VISUAL

Both hidden units receive the same pair $\boldsymbol{x} = (x_1, x_2)$:

$$h_1 = \mathrm{ReLU}(x_1 + x_2), \quad h_2 = \mathrm{ReLU}(x_1 + x_2 - 1)$$

all four input weights $= +1$



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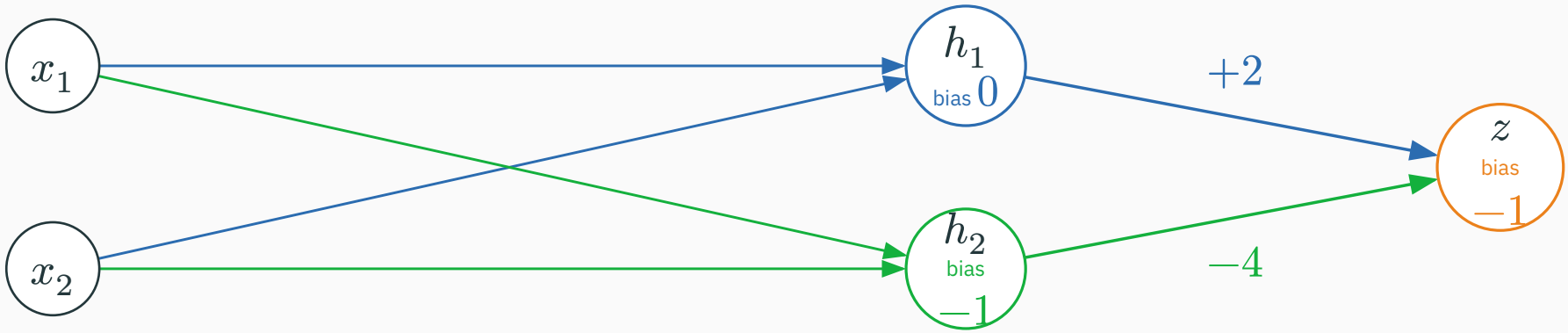

The output node computes $z = 2h_1 - 4h_2 - 1$; we keep this exact network throughout the XOR example.

Verify the XOR network on all four inputs

D DERIVATION

Input	$u_1 = x_1 + x_2$	h_1	$u_2 = x_1 + x_2 - 1$	h_2	$z = 2h_1 - 4h_2 - 1$	$\hat{y}$
$(0, 0)$	0	0	-1	0	-1	0
$(1, 0)$	1	1	0	0	1	1
$(0, 1)$	1	1	0	0	1	1
$(1, 1)$	2	2	1	1	-1	0

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$$\boldsymbol{W}_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{b}_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \boldsymbol{w}_{\text{out}} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}, \quad b_{\text{out}} = -1.$$

Verify the XOR network on all four inputs

D DERIVATION

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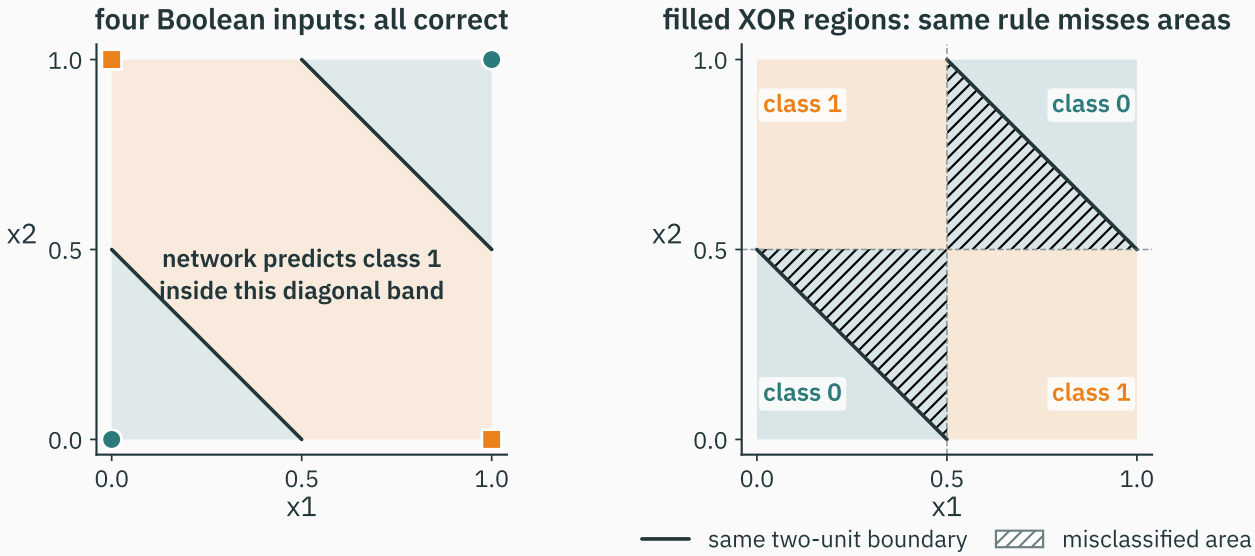
Predict 1 when $z \geq 0$. One ordinary output node correctly classifies all four XOR inputs.

Four-point XOR and filled XOR are different tasks

V VISUAL

The same hand-chosen logit is positive only in the band $0.5 \leq x_1 + x_2 \leq 1.5$.



Exact on the four Boolean inputs is not exact on filled regions. Audit both constructions in the XOR Colab ↗

Interactive: four points or four regions?

I INTERACTIVE

INTERACTIVE

↗ nipunbatra.github.io/interactive-articles/mlp-decision-boundary

Open the **ReLU classification lab**. Keep the model fixed while you switch from the four XOR corners to filled XOR.

1 · COMPARE THE DATA

four XOR corners → filled XOR

2 · COMPARE THE RULES

two-ReLU corner rule → four-ReLU field rule

3 · INSPECT THE FUNCTION

logit map, signed ReLU terms, and a rotatable 3-D surface

Then reset and train: representation tells us what is possible; optimization determines what is found.

First derive the feature functions for a general input

D DERIVATION

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. The weights and bias define a **function** before we substitute any particular input.

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BLUE HIDDEN UNIT

$$\boldsymbol{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad b_1 = 0$$

$$u_1(\boldsymbol{x}) = \boldsymbol{w}_1^\top \boldsymbol{x} + b_1$$
$$= 1x_1 + 1x_2 + 0 = x_1 + x_2$$

$$h_1(\boldsymbol{x})$$
$$= \mathrm{ReLU}(x_1 + x_2)$$

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GREEN HIDDEN UNIT

$$\boldsymbol{w}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad b_2 = -1$$

$$u_2(\boldsymbol{x}) = \boldsymbol{w}_2^\top \boldsymbol{x} + b_2$$
$$= 1x_1 + 1x_2 - 1 = x_1 + x_2 - 1$$

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First derive the feature functions for a general input

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$$h_2(\boldsymbol{x})$$
$$= \mathrm{ReLU}(x_1 + x_2 - 1)$$

These are exactly the two XOR features already shown in the node diagram; now we evaluate them at one input.

Check both hidden calculations in matrix form

D DERIVATION

$$\boldsymbol{u} = \boldsymbol{W}_1 \boldsymbol{x} + \boldsymbol{b}_1$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

$u_1 = 1(1) + 1(0) + 0 = 1$

$u_2 = 1(1) + 1(0) - 1 = 0$

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$$\boldsymbol{h} = \mathrm{ReLU}(\boldsymbol{u}) = \begin{pmatrix} \max(0,1) \\ \max(0,0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

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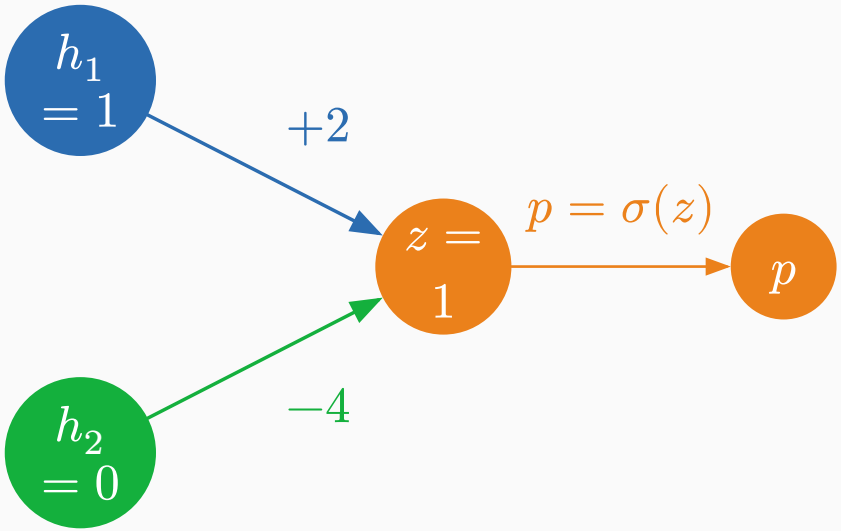
Rows of W_1 are neurons: each row computes one pre-activation, then ReLU acts elementwise.

Combine the two features, then compute probability and loss

D DERIVATION



Output logit

$$z = \begin{pmatrix} 2 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1$$

$$= 2(1) - 4(0) - 1 = 1.$$

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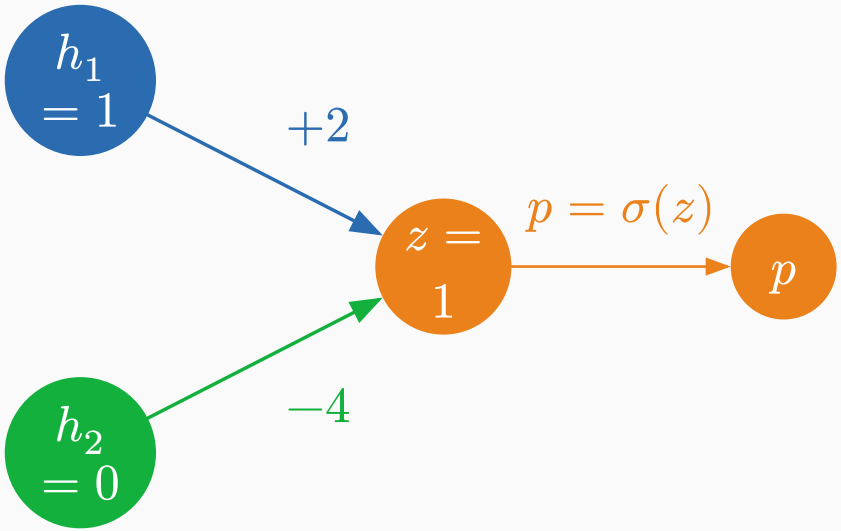
D DERIVATION



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$$p = \sigma(1) = \frac{1}{1 + \exp(-1)} \approx 0.731.$$

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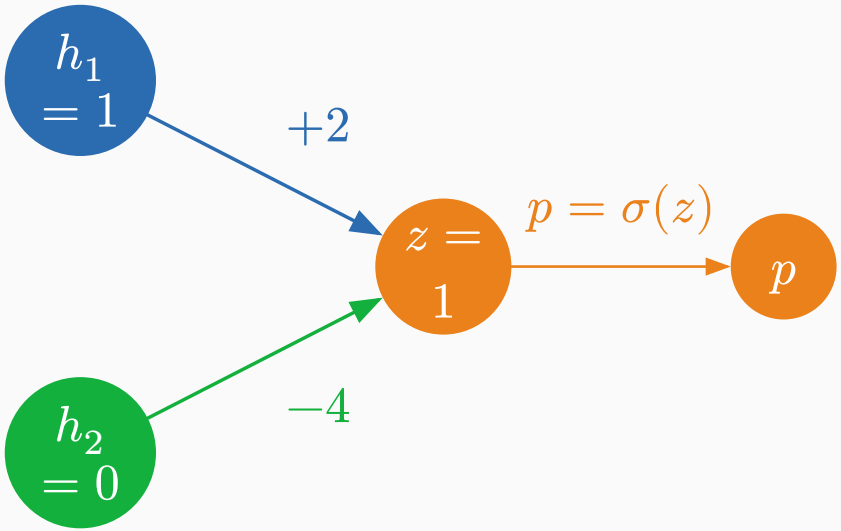

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Binary cross-entropy for $y = 1$

$$\mathcal{L} = -\log(0.731) \approx 0.313.$$

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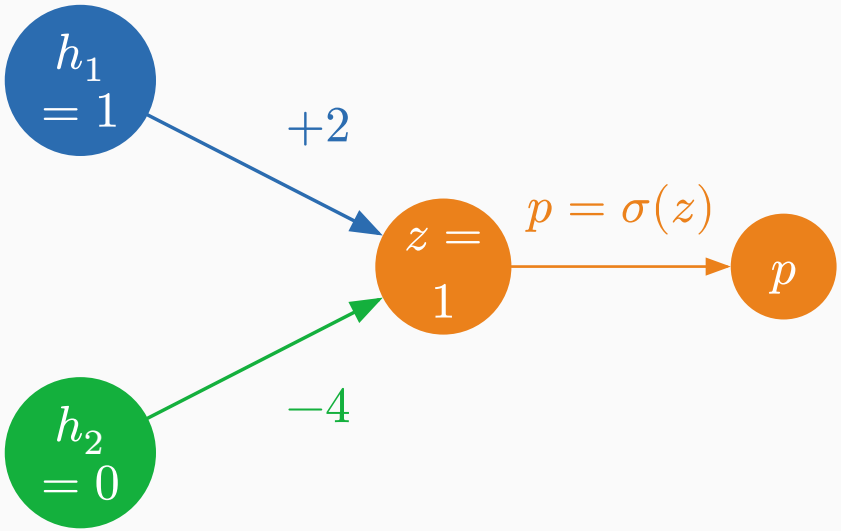

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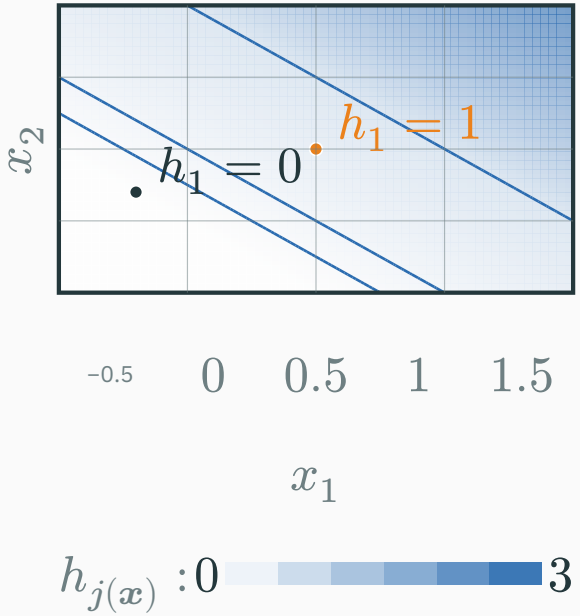
This is the same output node shown in the XOR network: weights $+2, -4$ and bias -1. Sigmoid only turns z into p.

Each ReLU is flat on one side of a line and rises on the other

V VISUAL

$$h_1(\boldsymbol{x}) = \mathrm{ReLU}(x_1 + x_2)$$

active above $x_1 + x_2 = 0$



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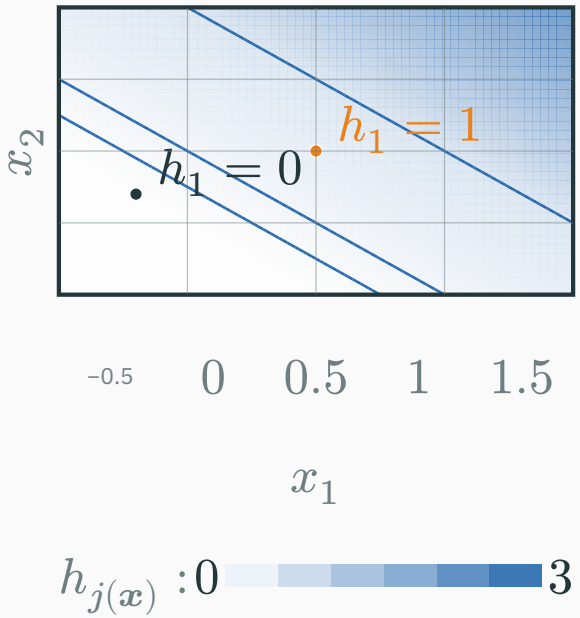
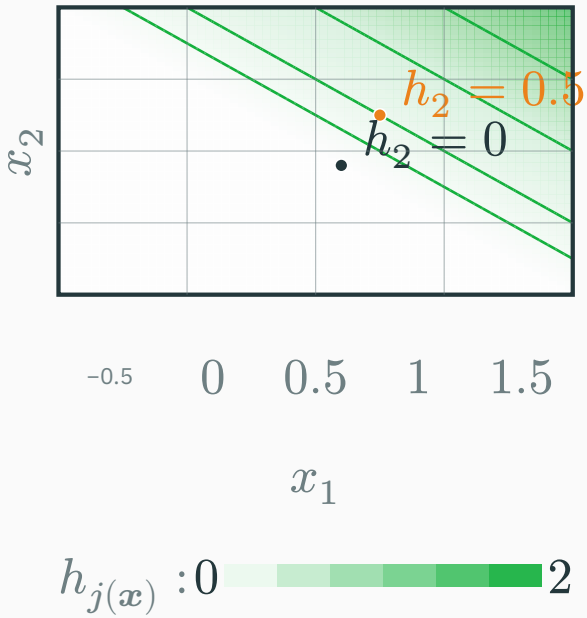


$h_2(\boldsymbol{x}) = \mathrm{ReLU}(x_1 + x_2 - 1)$

active above $x_1 + x_2 = 1$



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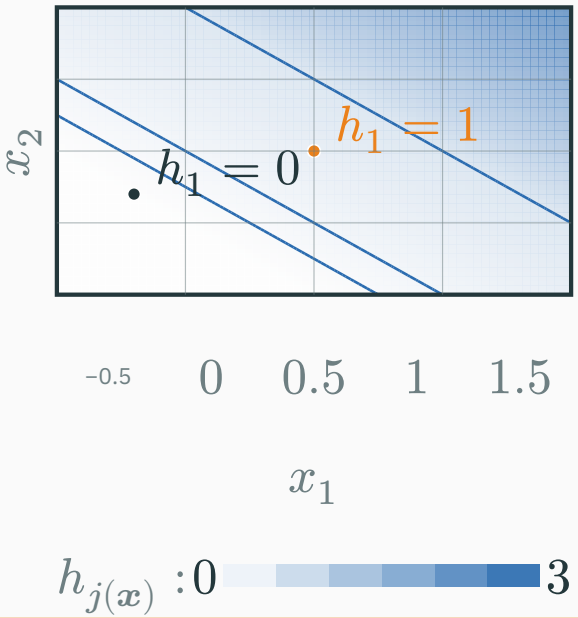

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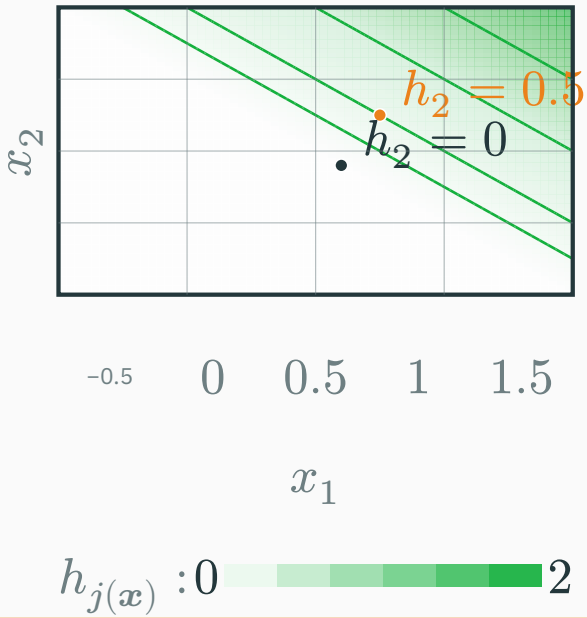


The grid is input space; each colour bar reports a hidden-unit activation value—not a probability.

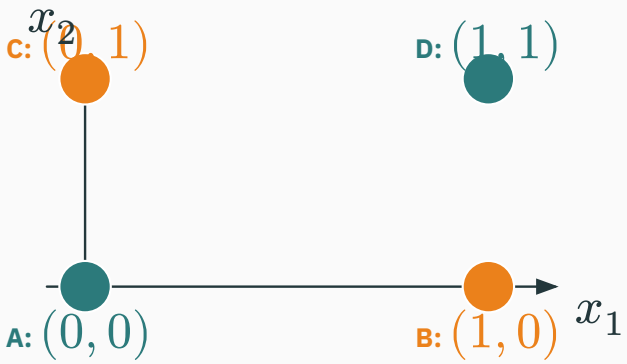
Compute each XOR point's new hidden-space coordinates

V VISUAL

Use the **same map** for every point: $T(x_1, x_2) = (h_1, h_2) = (\mathrm{ReLU}(x_1 + x_2), \mathrm{ReLU}(x_1 + x_2 - 1))$.

A: $(0,0) \to (0,0)$ **B:** $(1,0) \to (1,0)$ **C:** $(0,1) \to (1,0)$ **D:** $(1,1) \to (2,1)$

INPUT SPACE (x_1, x_2)



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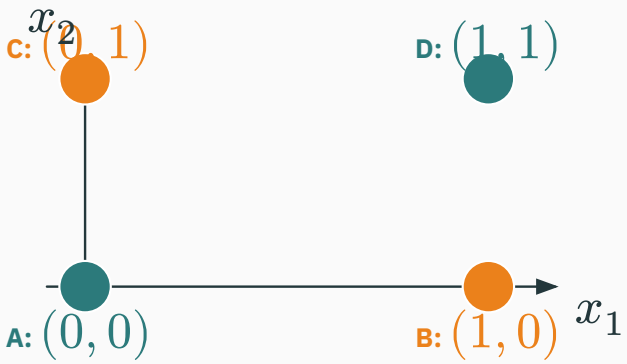

APPLY T

$\to$

B and C merge

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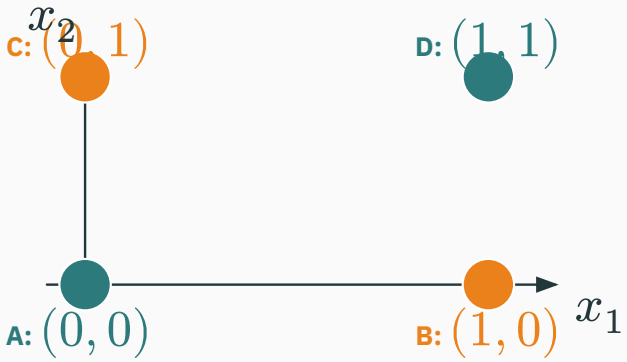
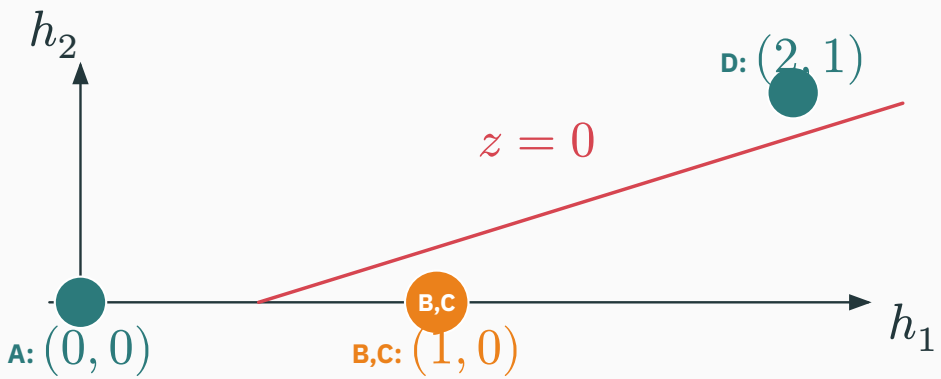


APPLY T

$\to$

B and C merge

HIDDEN SPACE (h_1, h_2)



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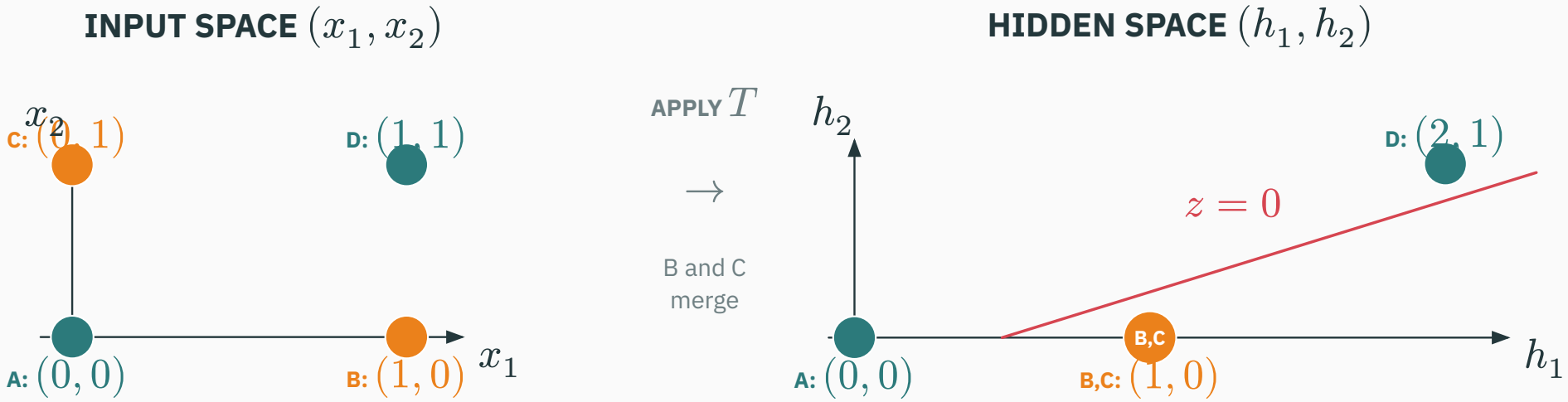
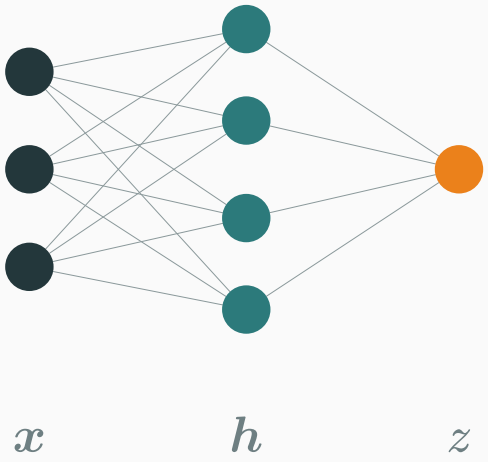


B and C both become $(1,0)$. Since $z = 2h_1 - 4h_2 - 1$, the boundary $z = 0 \leftrightarrow h_1 - 2h_2 = 0.5$ separates them from A and D.

Node view: binary uses one output; K-class uses K

V VISUAL

BINARY



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V VISUAL

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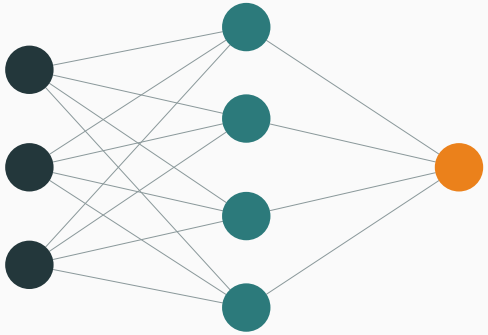
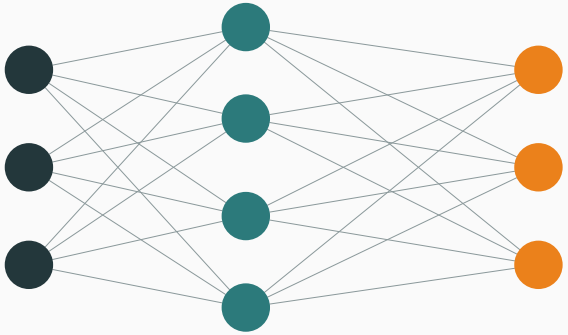


x h z

THREE CLASSES

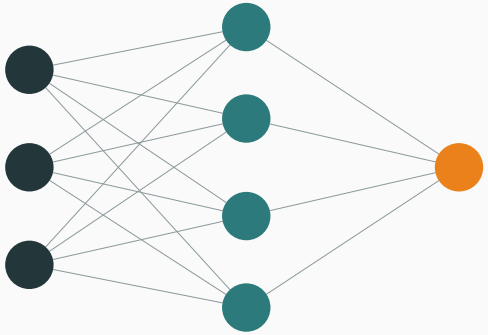


x h z_1, z_2, z_3

Node view: binary uses one output; K-class uses K

V VISUAL

BINARY



x h z

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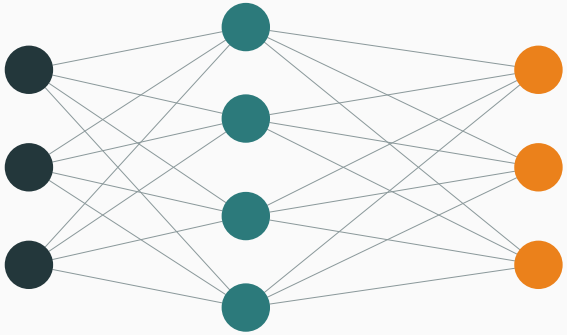
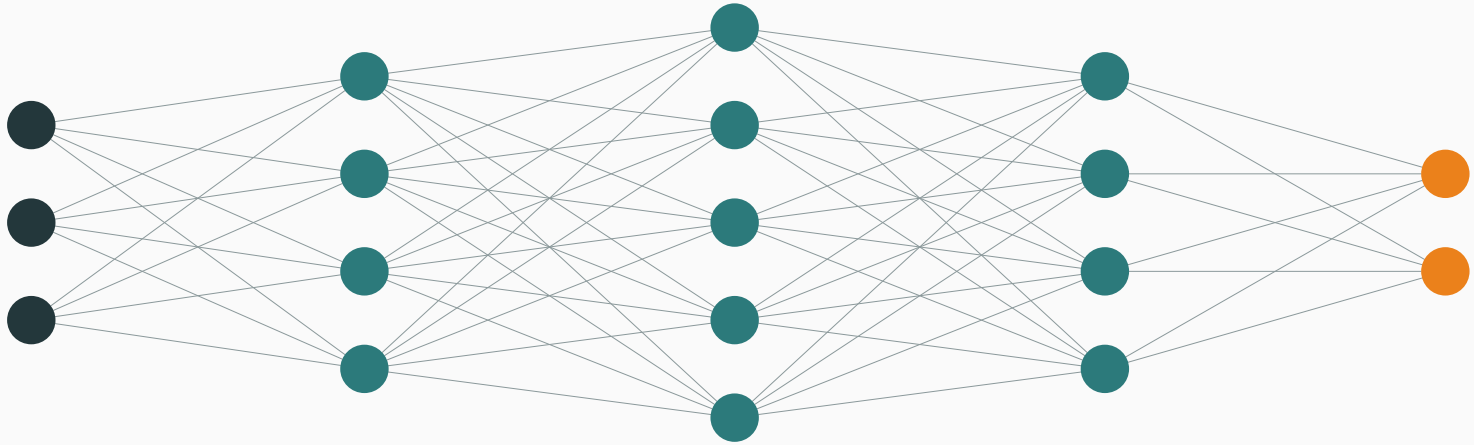


x h z_1, z_2, z_3

The hidden representation can stay the same; only the number and interpretation of output nodes changes.

Node view: depth adds hidden columns

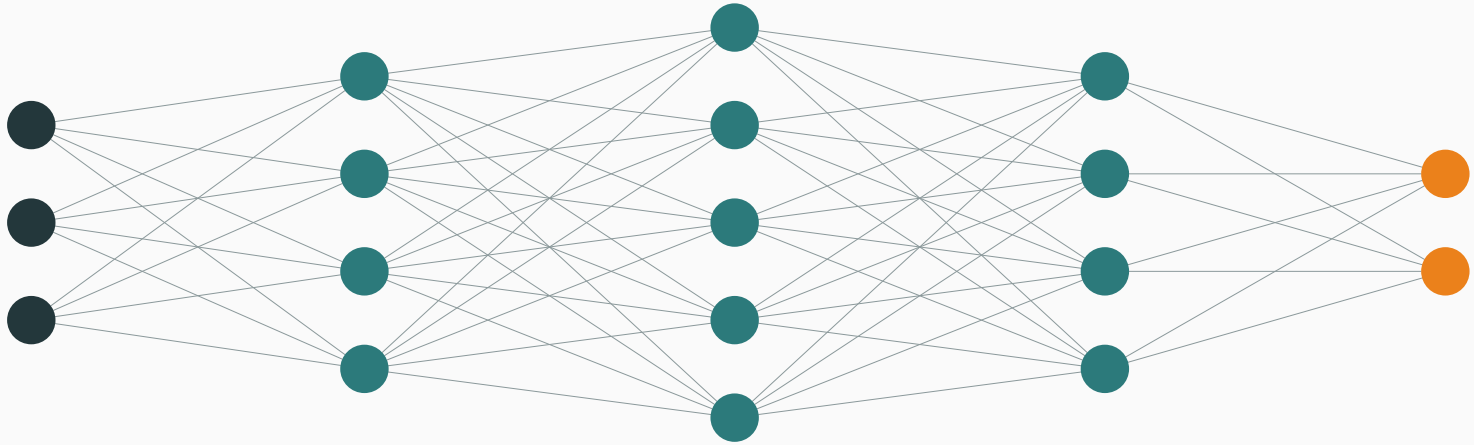
V VISUAL



input x hidden $h^{(1)}$ hidden $h^{(2)}$ hidden $h^{(3)}$ output z

Node view: depth adds hidden columns

V VISUAL

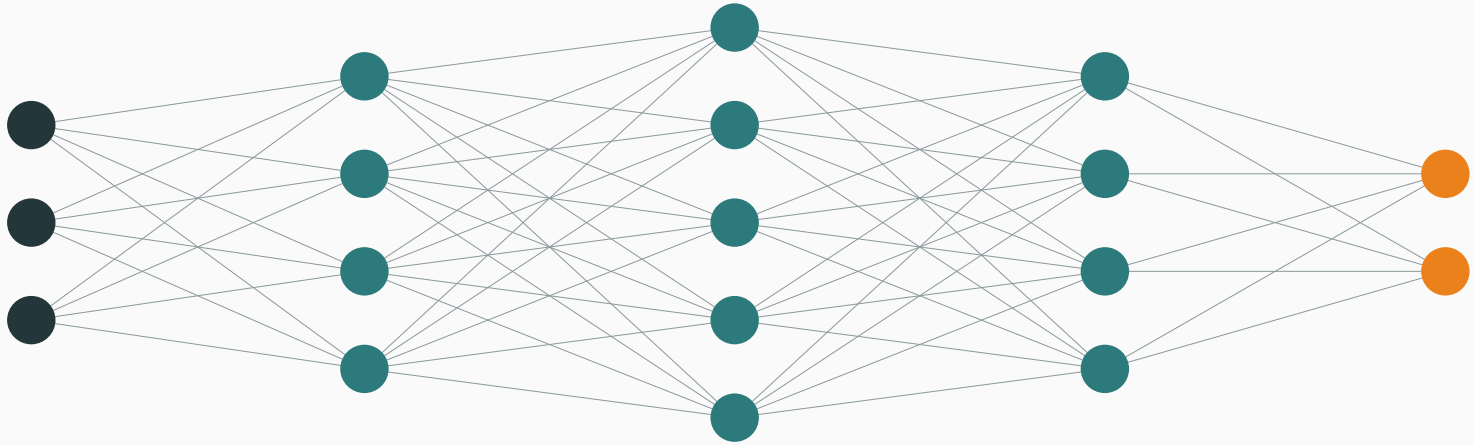


input x hidden $h^{(1)}$ hidden $h^{(2)}$ hidden $h^{(3)}$ output z

Every node connects only to the next column; information reaches the output through successive learned transformations.

Node view: depth adds hidden columns

V VISUAL



input x | hidden $h^{(1)}$ | hidden $h^{(2)}$ | hidden $h^{(3)}$ | output z

Every node connects only to the next column; information reaches the output through successive learned transformations.

Depth is visible as the number of hidden columns, not the number of nodes within one column.

Universal approximation

Universal approximation asks one precise question

Q QUESTION

TARGET
continuous f

$\rightarrow$

REGION
closed, bounded D

$\rightarrow$

TOLERANCE
$\varepsilon > 0$

Universal approximation asks one precise question

Q QUESTION

TARGET continuous f → **REGION** closed, bounded D → **TOLERANCE** $\varepsilon > 0$

Can some finite network g make $|f(x) - g(x)| < \varepsilon$ for **every** $x \in D$?

Universal approximation asks one precise question

Q QUESTION

TARGET		REGION		TOLERANCE
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With a suitable nonlinear activation, the theorem says: yes—such a finite one-hidden-layer network exists.

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With a suitable nonlinear activation, the theorem says: yes—such a finite one-hidden-layer network exists.

This is an existence claim about representational capacity; it does not promise that training will find the network.

A ReLU contributes one hinge

V VISUAL

$$\varphi(z) = \max(0, z)$$

A ReLU contributes one hinge

V VISUAL

$$\varphi(z) = \max(0, z)$$

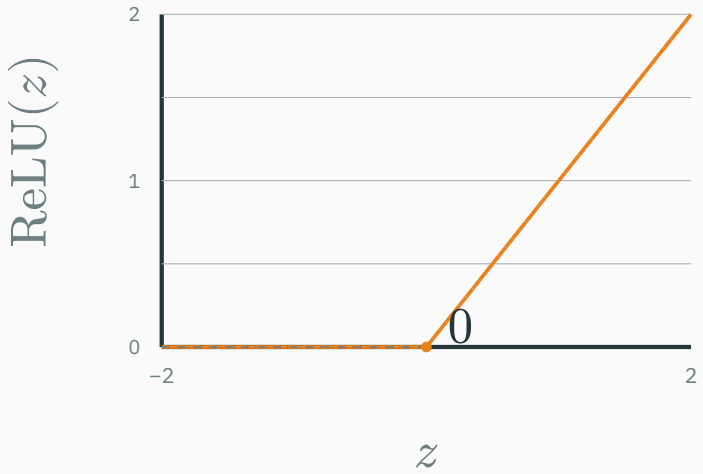
- $z < 0$: output is zero;
- $z > 0$: output grows linearly;
- at $z = 0$: the slope changes—a **hinge**.

A ReLU contributes one hinge

V VISUAL

$$\varphi(z) = \max(0, z)$$

- $z < 0$: output is zero;
- $z > 0$: output grows linearly;
- at $z = 0$: the slope changes—a **hinge**.



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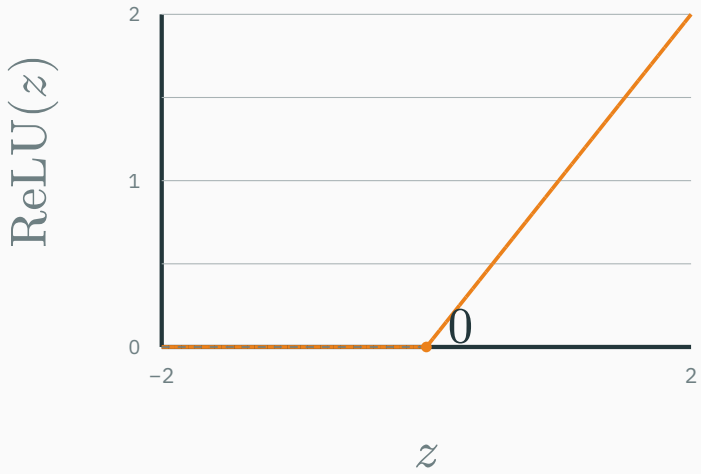
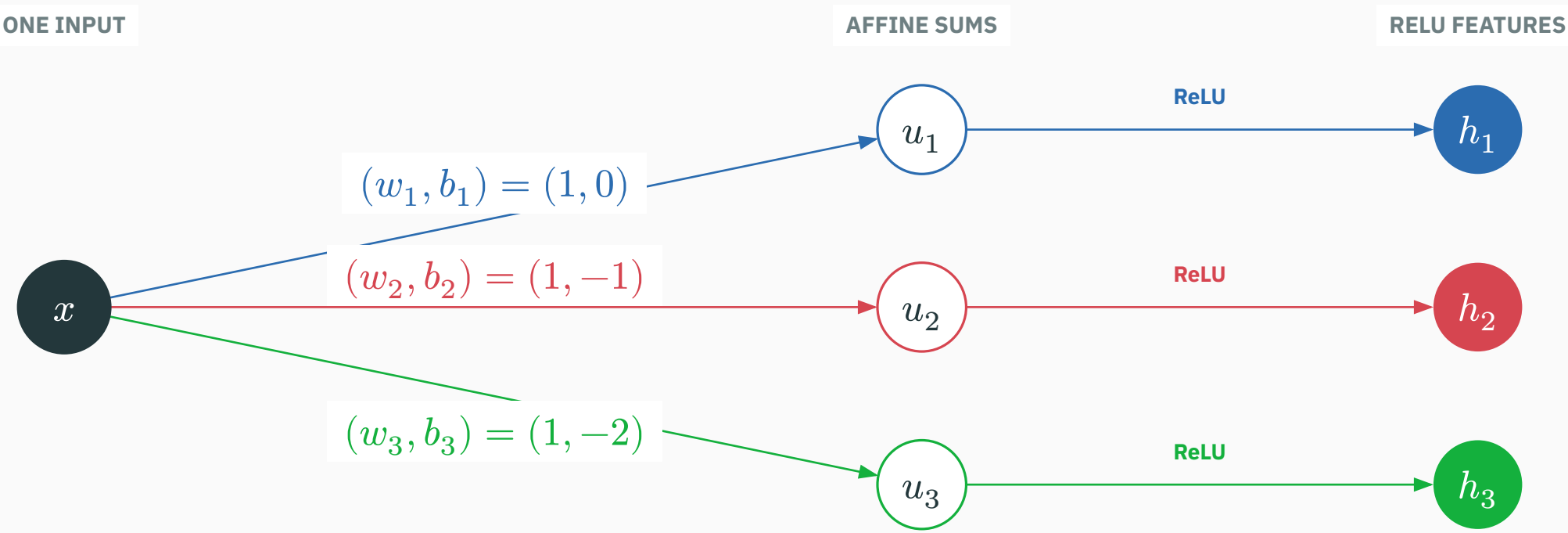


A hidden layer combines many shifted hinges into a piecewise-linear function.

One input becomes three symbolic hinge features

V VISUAL

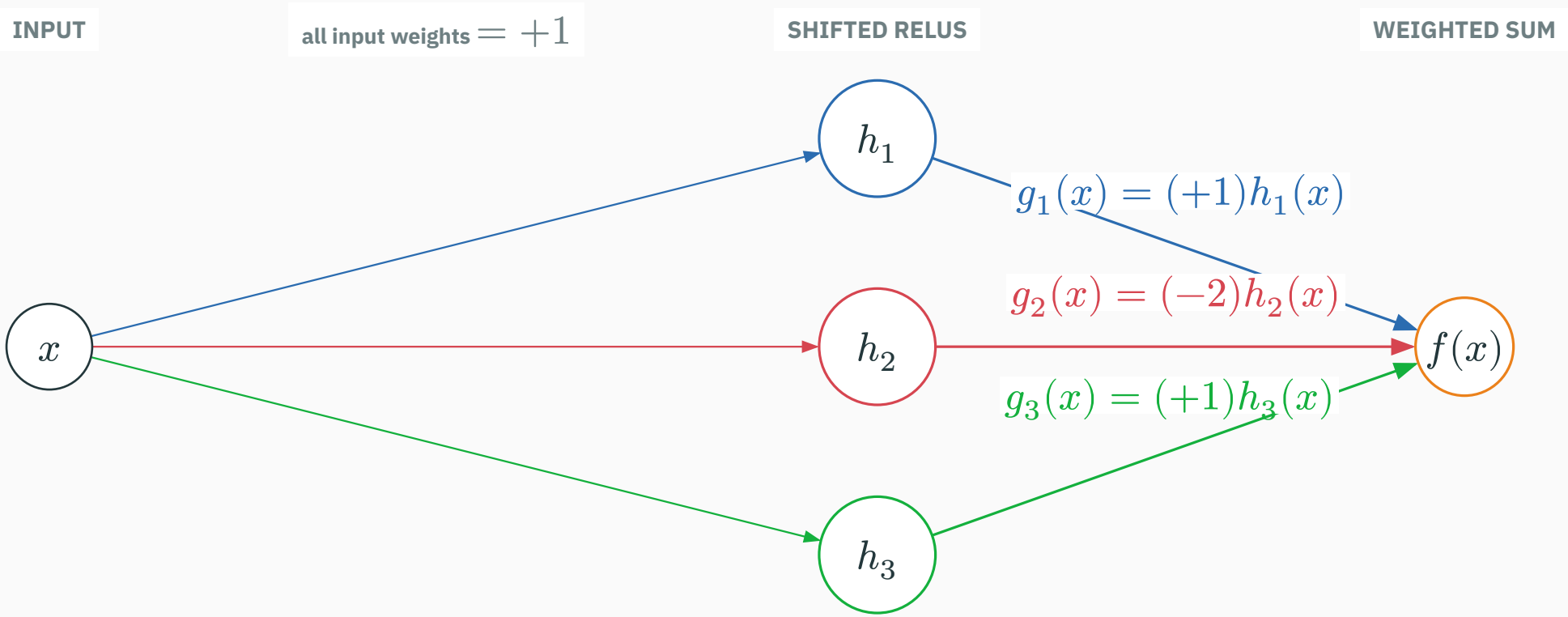
Every unit first computes $u_i = w_i x + b_i$, then applies $h_i = \mathrm{ReLU}(u_i)$.



The same x creates $h(x) = (\mathrm{ReLU}(x), \mathrm{ReLU}(x-1), \mathrm{ReLU}(x-2))$; the hinges are at $0, 1, 2$.

Output weights combine the three symbolic hinges

V VISUAL



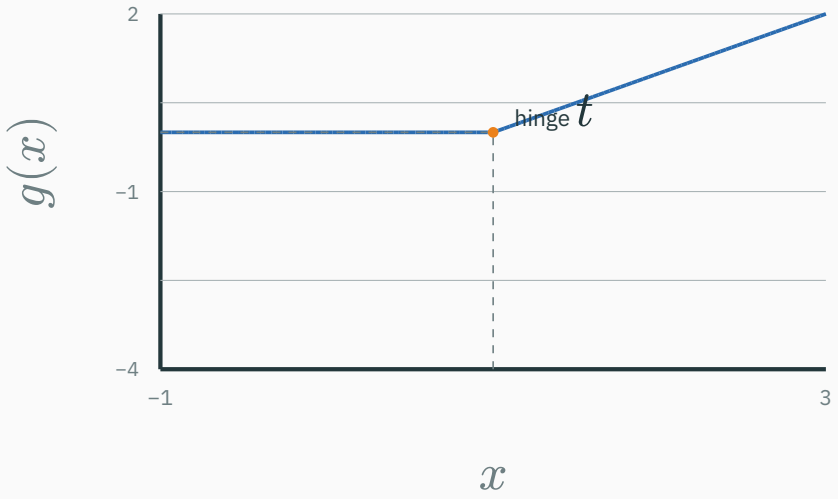
The output adds $f(x) = g_1(x) + g_2(x) + g_3(x)$—the theorem's "sum of hidden units" is literal.

Output weights choose each hinge's slope change

V VISUAL

Start with $h(x) = \mathrm{ReLU}(x - t)$; its output contribution is $g(x) = vh(x)$.

$v = +1 \quad \rightarrow \quad g(x) = +h(x)$



slope change $\Delta s = +1$

$v = -2 \quad \rightarrow \quad g(x) = -2h(x)$

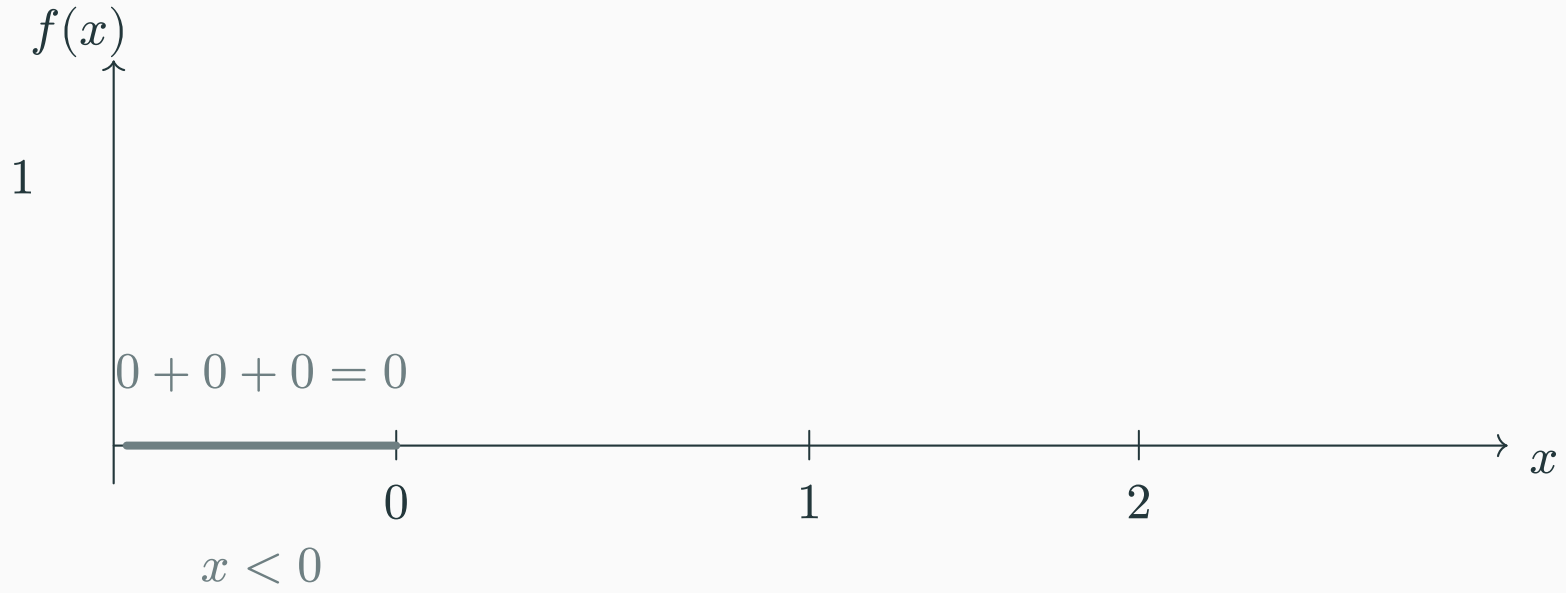
slope change $\Delta s = -2$

The hinge stays at t; the sign of v chooses the bend, and $|v|$ sets its strength.

Three hinges assemble a tent interval by interval

D DERIVATION

$$f(x) = h_1 - 2h_2 + h_3 = \mathrm{ReLU}(x) - 2\mathrm{ReLU}(x-1) + \mathrm{ReLU}(x-2).$$



Region 1: every hidden unit is off, so $f(x) = 0$.

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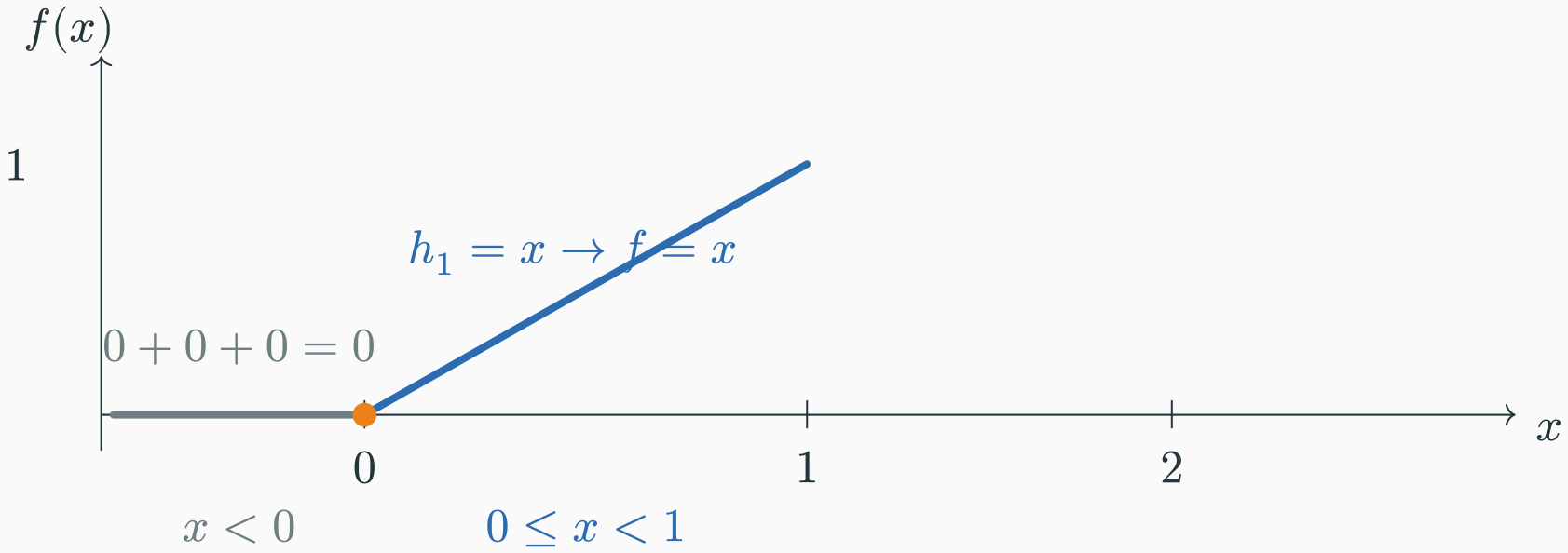
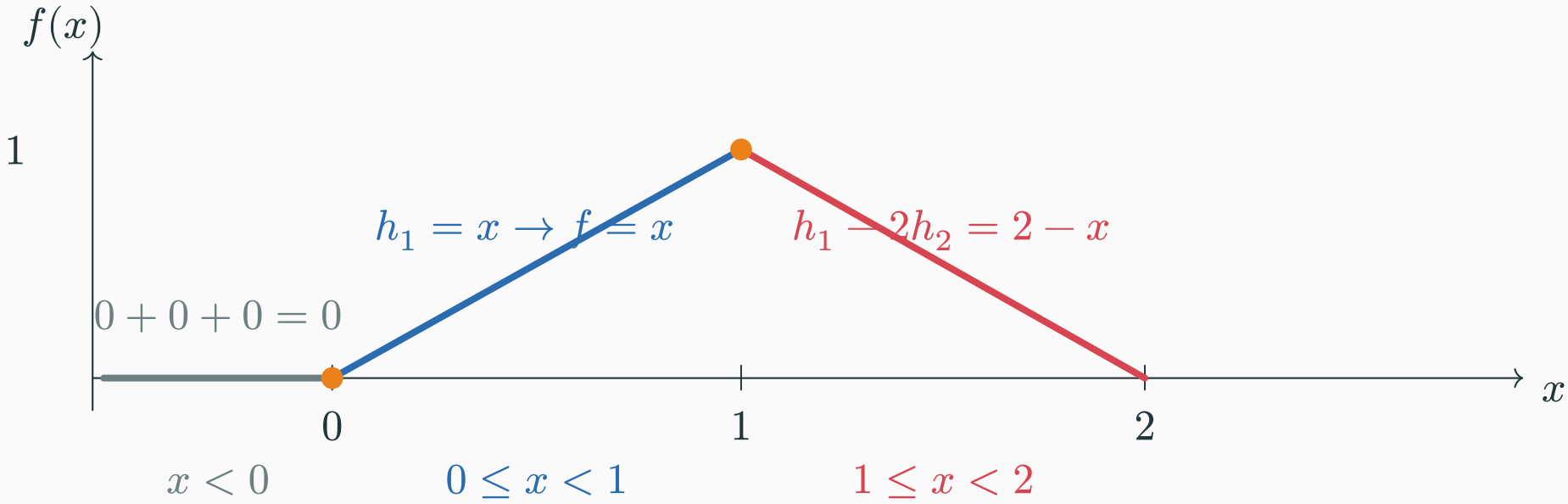


Region 2: $h_1 = x$ switches on, so the output rises with slope $+1$.

Three hinges assemble a tent interval by interval

D DERIVATION

$$f(x) = h_1 - 2h_2 + h_3 = \mathrm{ReLU}(x) - 2\mathrm{ReLU}(x-1) + \mathrm{ReLU}(x-2).$$



Region 3: $-2h_2$ switches on, changing the total slope from $+1$ to -1.

Three hinges assemble a tent interval by interval

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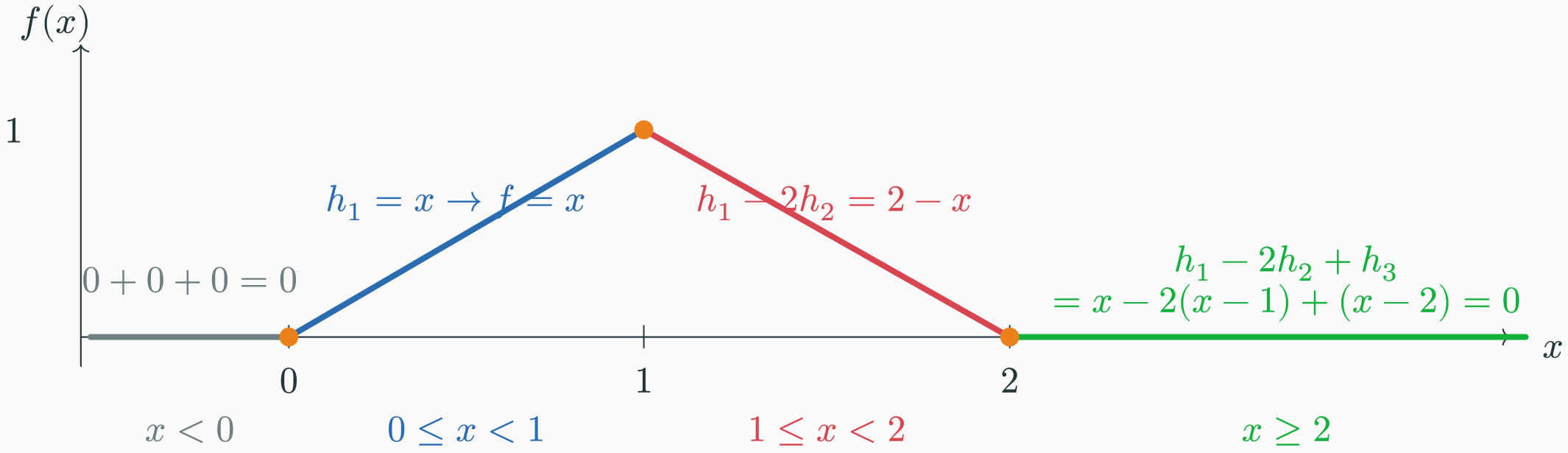

Region 4: h_3 switches on and cancels the remaining slope, so $f(x) = 0$.

Reading left to right: flat → rise → fall → flat—the weighted sum is a tent.

Learned ReLU units are the theorem's building blocks

D DERIVATION

For a one-dimensional input, a trained hidden unit has the form

$$h_{j(x)} = \mathrm{ReLU}\big(w_j x + b_j\big), \quad g(x) = c + \sum_j v_j h_{j(x)}.$$

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j	hinge t_j	w_j	b_j	output v_j	signed contribution $g_{j(x)}$
1	0	1	0	$+1$	$+\mathrm{ReLU}(x)$
2	1	1	-1	-2	$-2\mathrm{ReLU}(x-1)$
3	2	1	-2	$+1$	$+\mathrm{ReLU}(x-2)$

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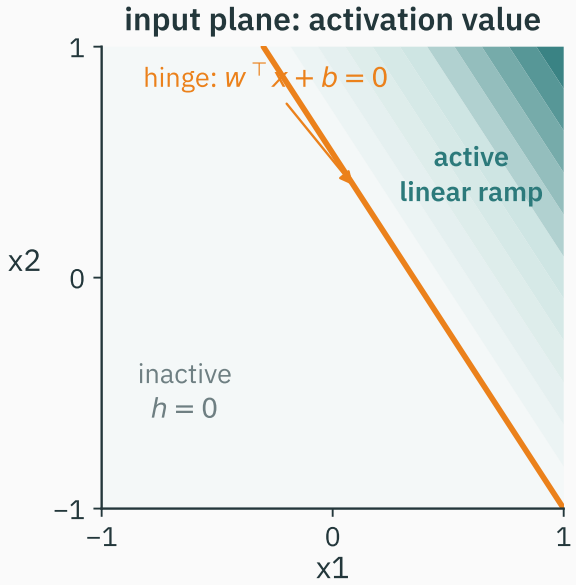
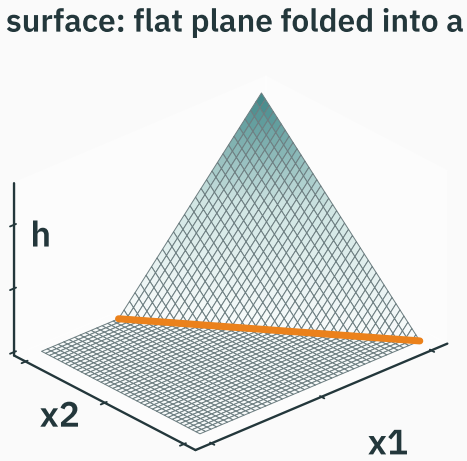
Training learns hinge locations and signed contributions; universal approximation says enough such units can make the remaining gap arbitrarily small.

One ReLU folds a 2-D input plane into a 3-D surface

V VISUAL

$$h(\boldsymbol{x}) = \mathrm{ReLU}(\boldsymbol{w}^\top \boldsymbol{x} + b) = \mathrm{ReLU}(x_1 + 0.65x_2 - 0.35)$$



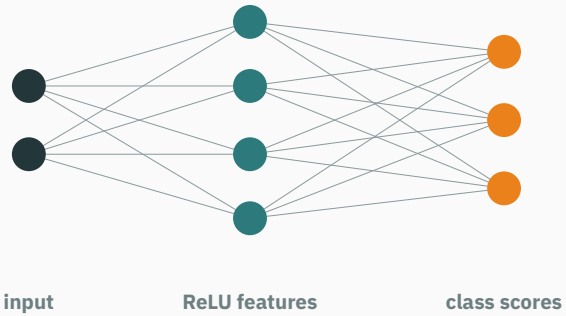


The zero line is the hinge: one side stays flat, the other rises linearly. More units add folds in different positions and directions.

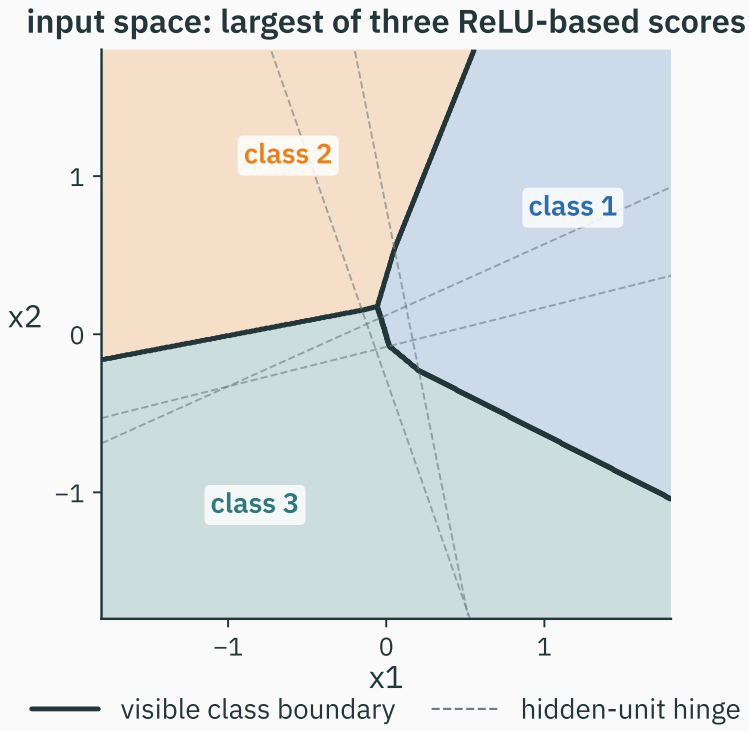
ReLU features create piecewise-linear class boundaries

V VISUAL

FEATURES FEED CLASS SCORES



$$h_{j(\boldsymbol{x})} = \mathrm{ReLU}\left(\boldsymbol{w}_j^\top \boldsymbol{x} + b_j\right)$$
$$z_k = c_k + \sum_j v_{kj} h_{j(\boldsymbol{x})}$$
$$\hat{y} = \arg\max_k z_k$$



Inside one activation pattern, pairwise score ties are straight; across ReLU hinges they join into piecewise-linear boundaries in input space.

Interactive: build or learn a function

I INTERACTIVE

INTERACTIVE

↗ nipunbatra.github.io/interactive-articles/relu-function-approximation

Open the **ReLU function lab**. Construction and training answer different questions, so the lab keeps them in separate lanes.

CONSTRUCT

choose a target and drag the hinge locations

TRAIN

keep the width fixed, reset the seed, and optimize

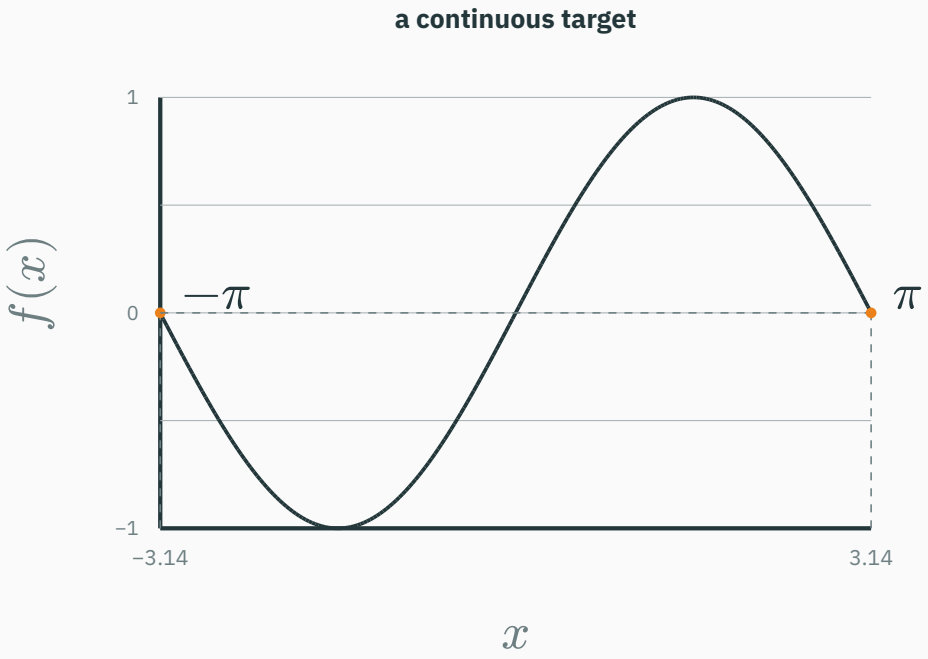
INSPECT

learned function, residual, grid gap, and signed ReLU terms

More hinges increase representational freedom; the residual shows what the chosen construction or training run still misses.

First fix the target and the input region

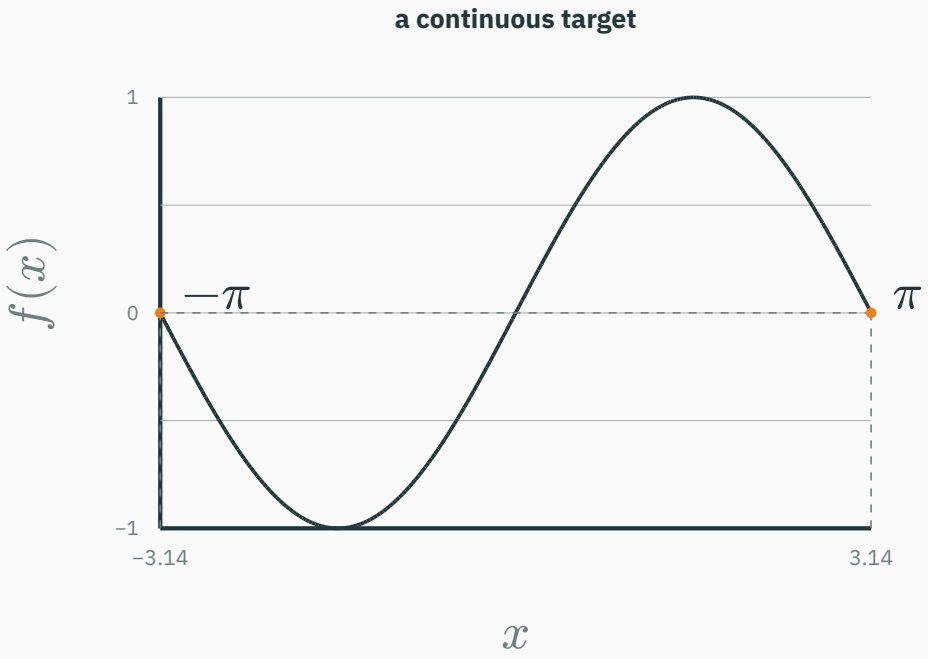
V VISUAL



Continuous means that the target has no jumps.

First fix the target and the input region

V VISUAL



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In one dimension, a **compact domain** can be a closed, bounded interval such as

$$D = [-\pi, \pi].$$

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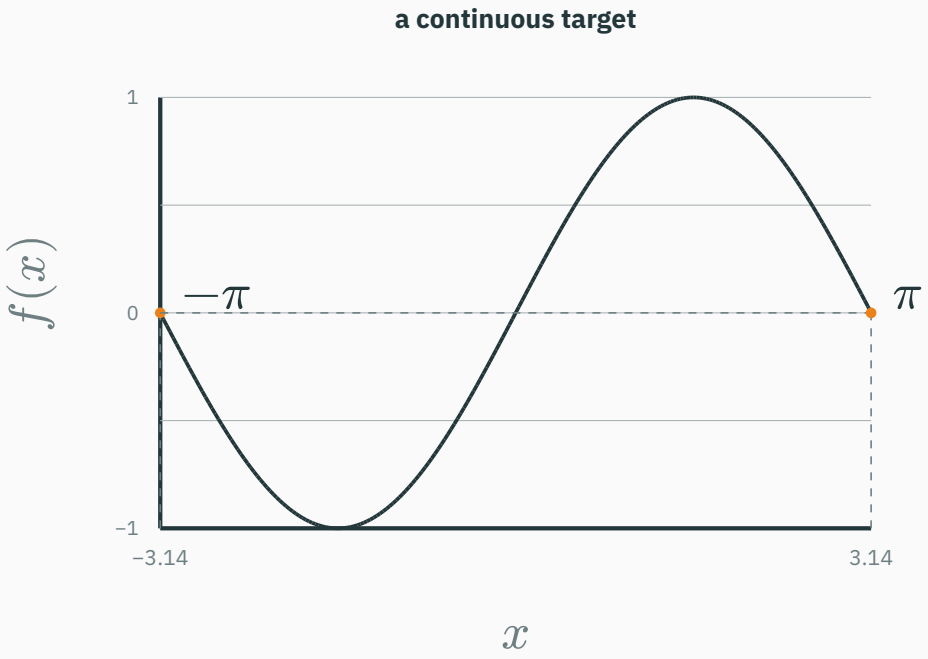
V VISUAL



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We ask the network to match f **on** D; the theorem makes no claim outside that chosen region.

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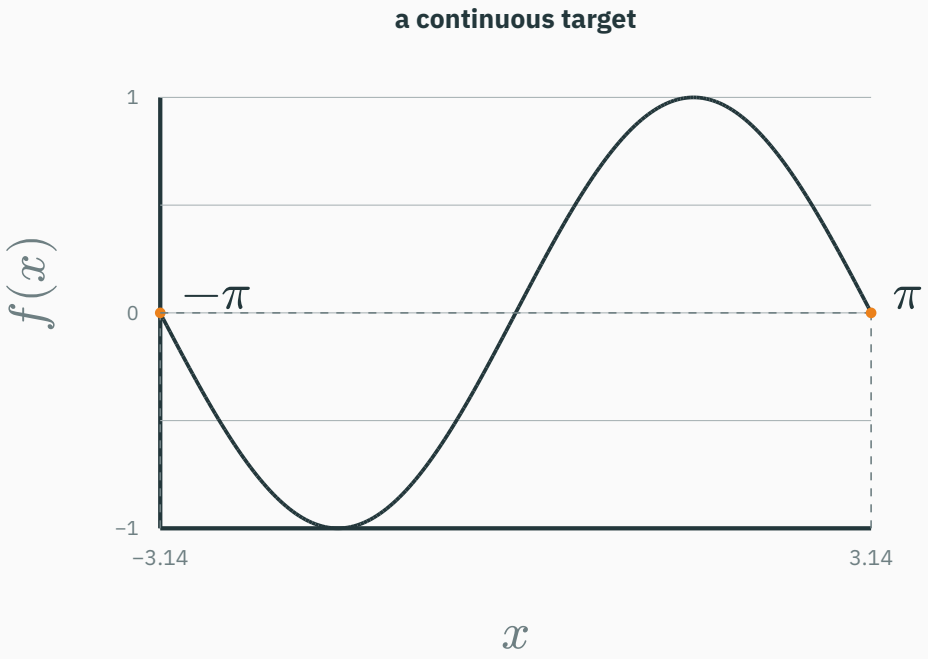

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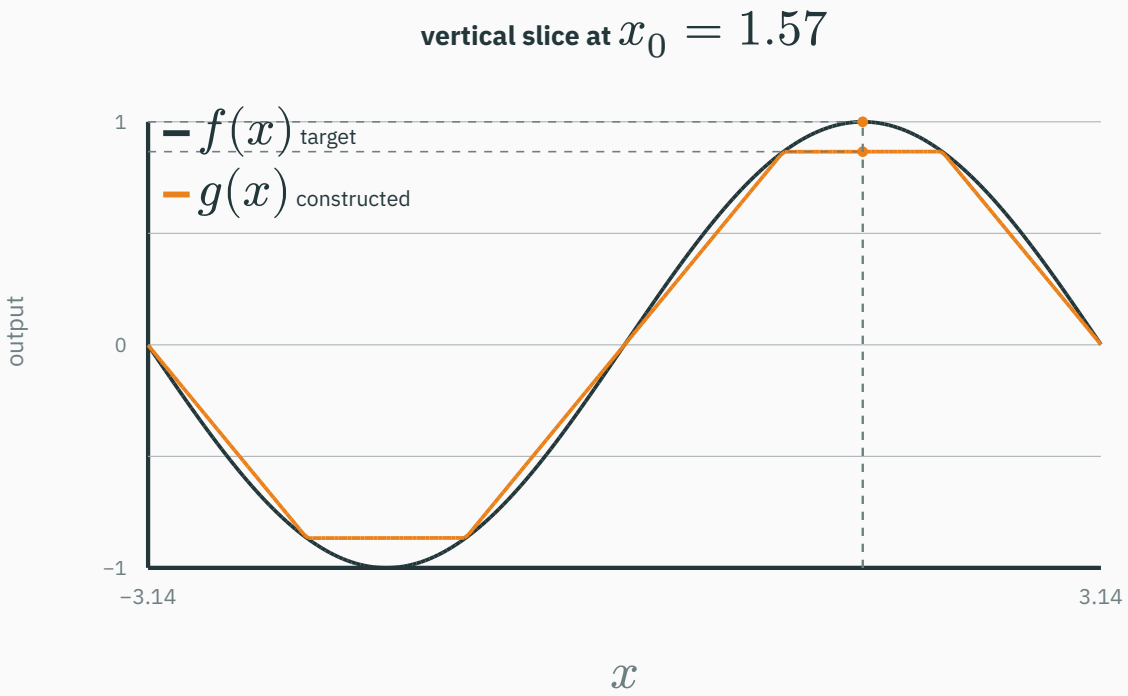
Approximation begins with one target function on one specified input region.

At one input, error is one vertical gap

V VISUAL

$$e(x_0) = |f(x_0) - g(x_0)|$$

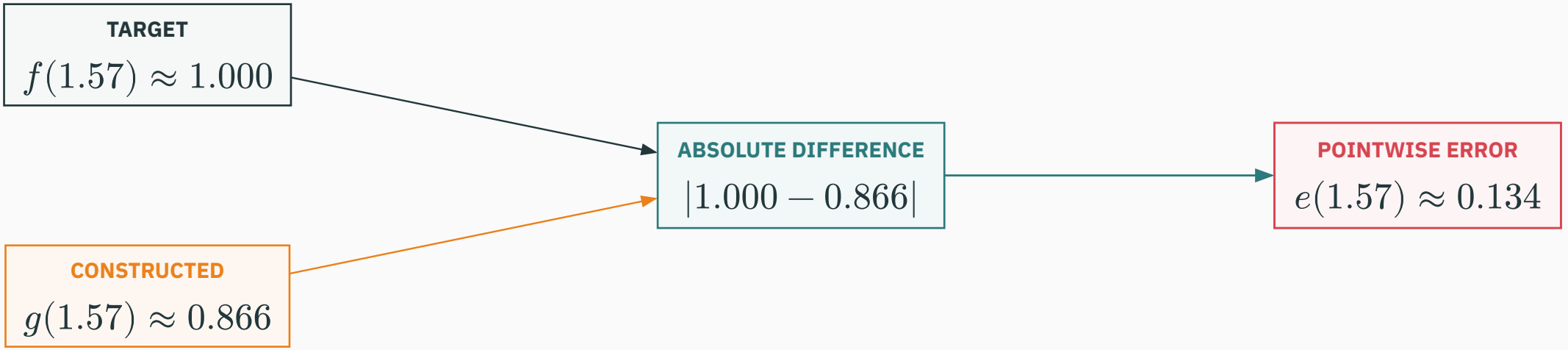


Pointwise error is the vertical distance between the target and its constructed approximation at the same input.

Calculate the gap at $x_0 = 1.57$

D DERIVATION

$$e(x_0) = |f(x_0) - g(x_0)|$$



One point is not enough: universal approximation asks for the largest gap over the whole region.

“sup” means the worst gap over the whole region

V VISUAL

$\sup_{x \in D} |f(x) - g(x)|$

sup means **supremum**: the smallest number at least as large as every pointwise error.

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Here the error is continuous and $D = [-\pi, \pi]$ is closed and bounded, so a highest error exists. In this plot, read **sup** as **maximum**.

$\sup < \varepsilon$: no input in D crosses the tolerance.

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The entire orange error curve stays below $\varepsilon = 0.05$, so the uniform guarantee is satisfied.

More hinges reduce the largest approximation error

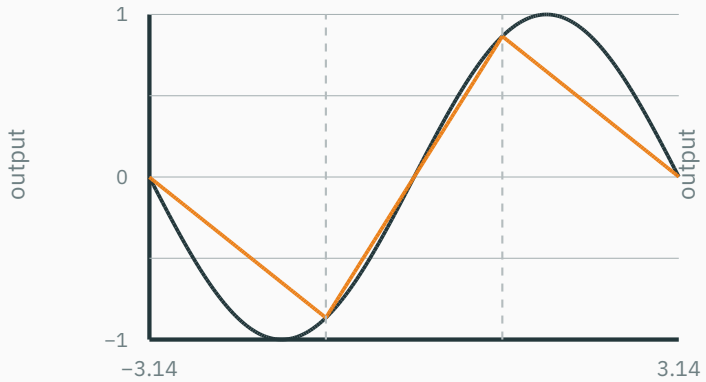
V VISUAL

CONSTRUCTED INTERPOLANTS · FIXED KNOTS · NOT TRAINED solid: target $\sin x$ dashed: $g_{m(x)}$

Each g_m matches $\sin x$ at fixed knots and is linear between adjacent knots.

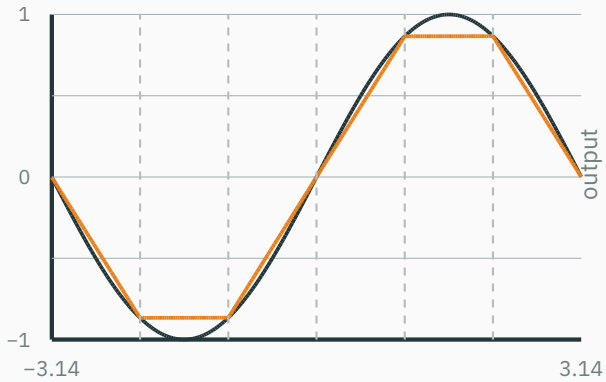
2 hinges

$\varepsilon_m \approx 0.437$



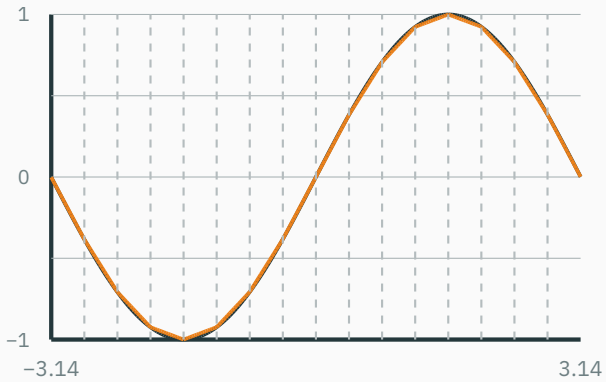
5 hinges

$\varepsilon_m \approx 0.134$



15 hinges

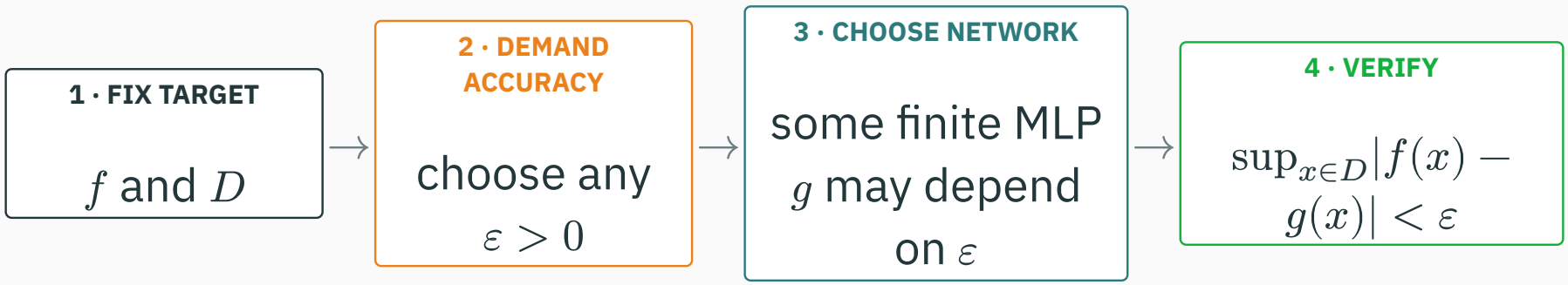
$\varepsilon_m \approx 0.019$



More hinges shorten each linear segment, so the worst gap falls from 0.437 to 0.019.

Read the universal-approximation claim in order

D DERIVATION



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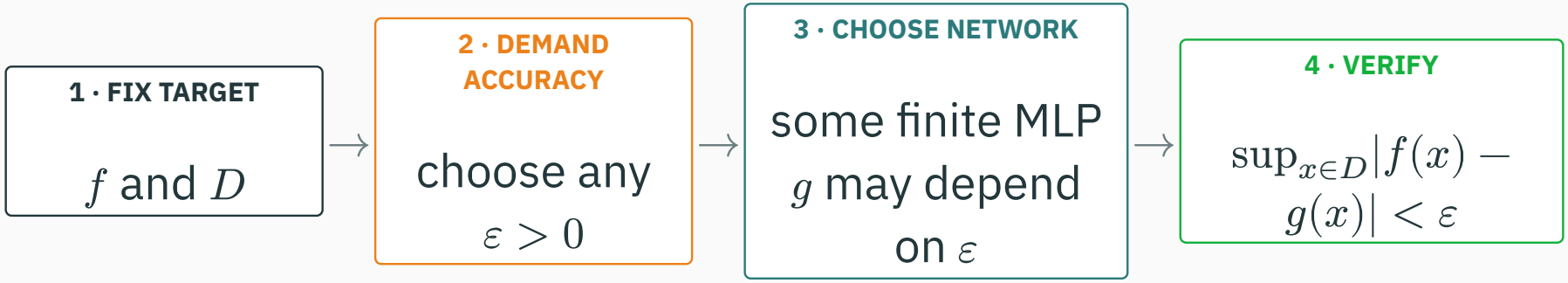

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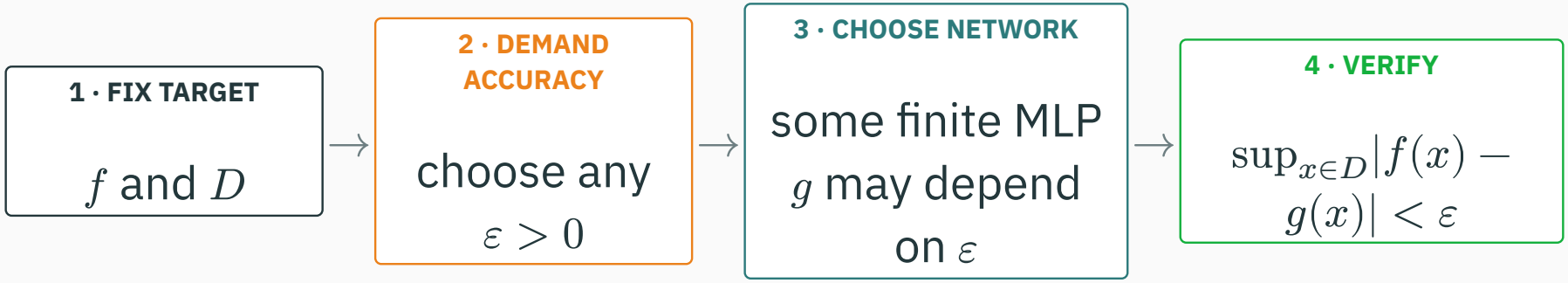

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For every ε: we may request any positive tolerance.

There exists g: the network may change when the tolerance changes.

Read the universal-approximation claim in order

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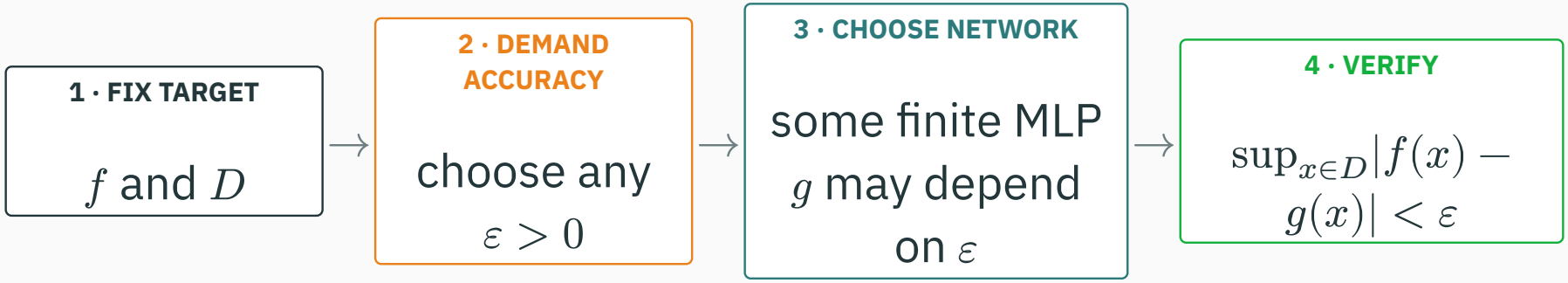

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For every ε: we may request any positive tolerance.

There exists g: the network may change when the tolerance changes.

This is a statement about representational capacity—not yet about how to find the network.

Existence does not guarantee learnability

Universal approximation does **not** promise:

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- that training will **find** that function;
- that **finite data** is enough;

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- that a **small** network suffices;
- that it will **generalize**.

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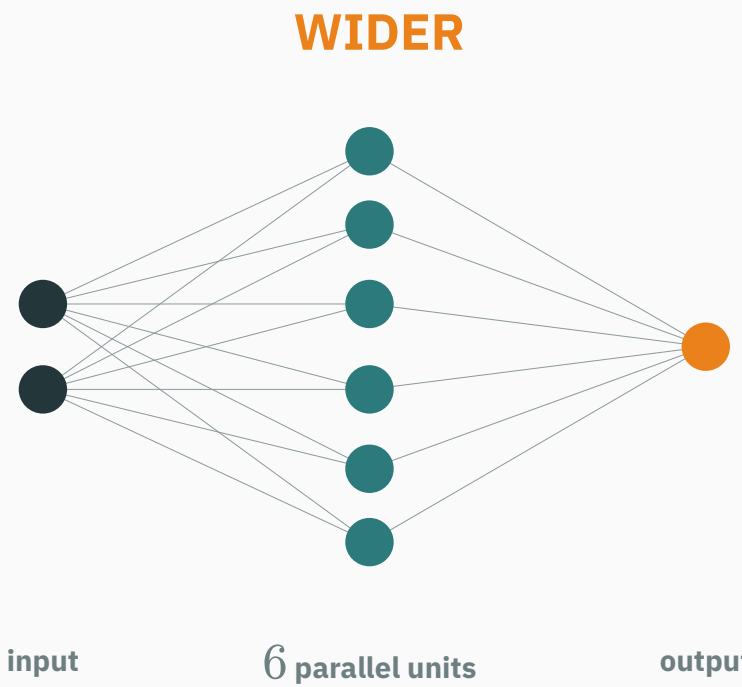
expressivity $\neq$ trainability $\neq$ generalization

Depth versus width

Node view: width adds rows; depth adds columns

V VISUAL



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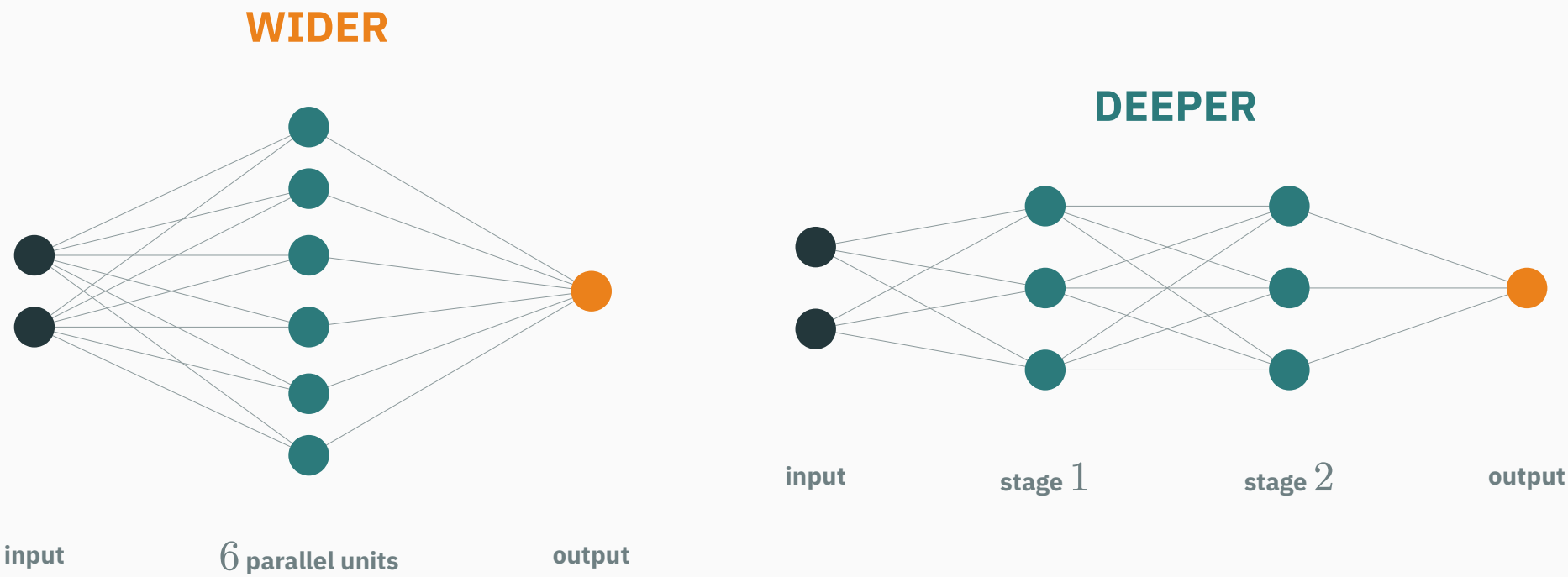

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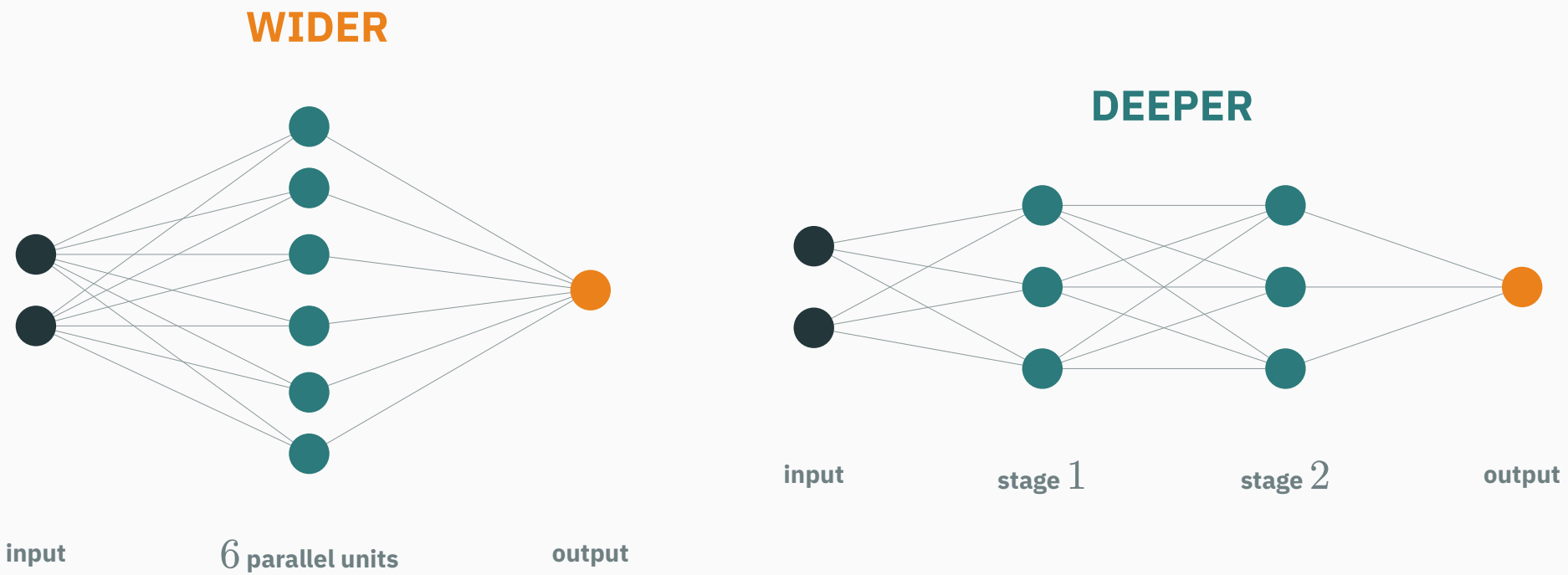

Count nodes vertically to read width; count hidden columns horizontally to read depth.

Depth means composition: layer 2 receives h_1, not x

V VISUAL

$$x \to h_1(x) = \mathrm{ReLU}(2x + 1) \to h_2(h_1) = \mathrm{ReLU}(2 - h_1)$$

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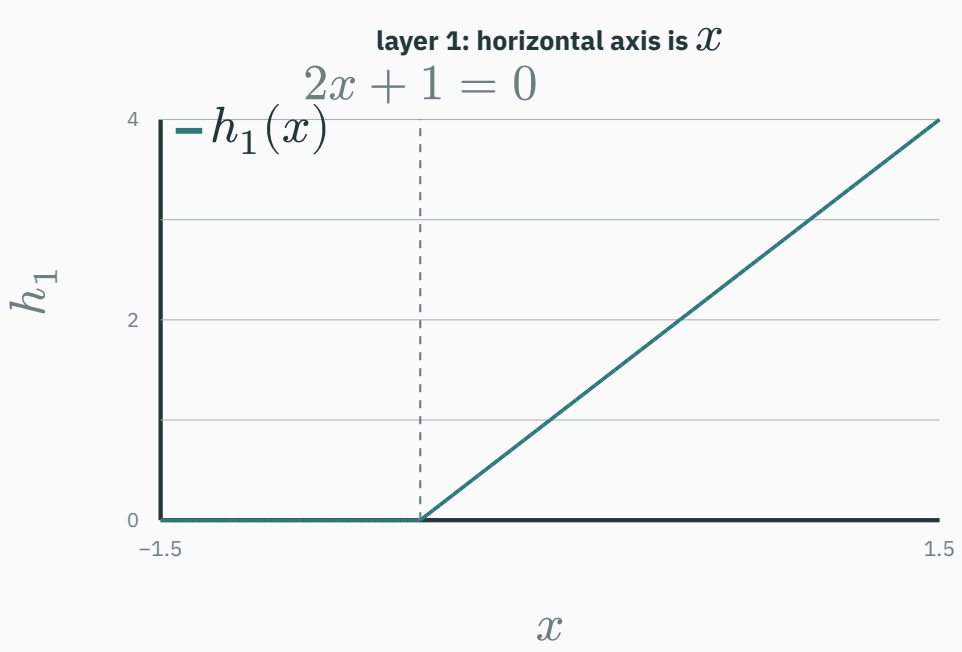

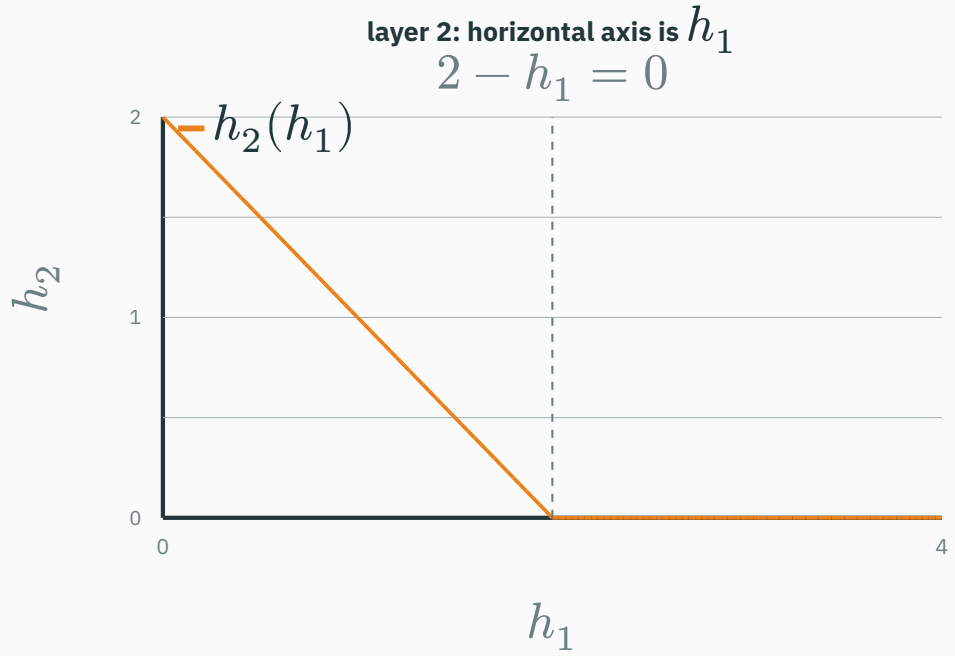

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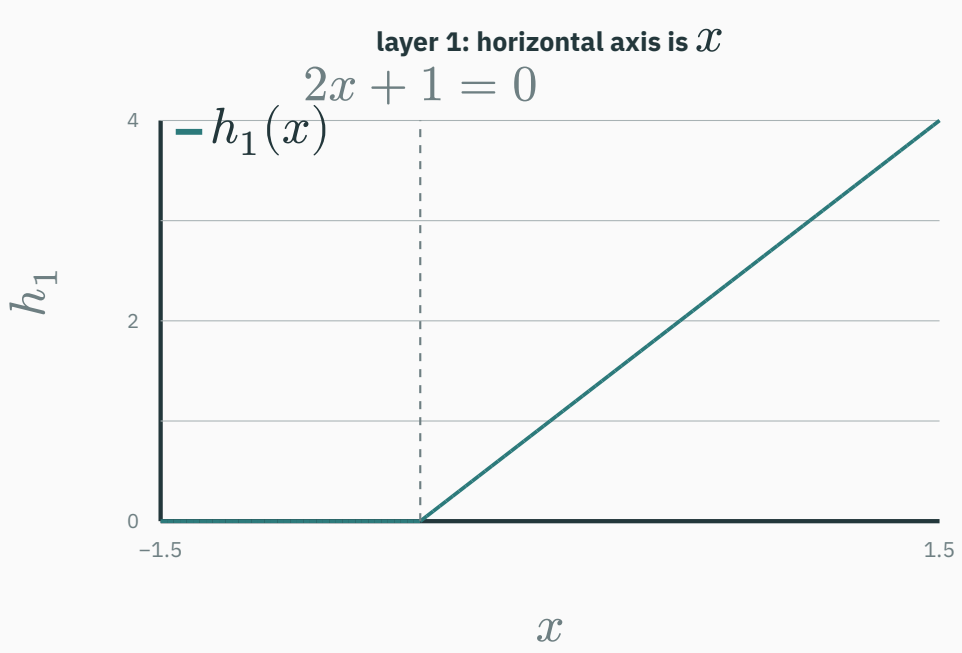

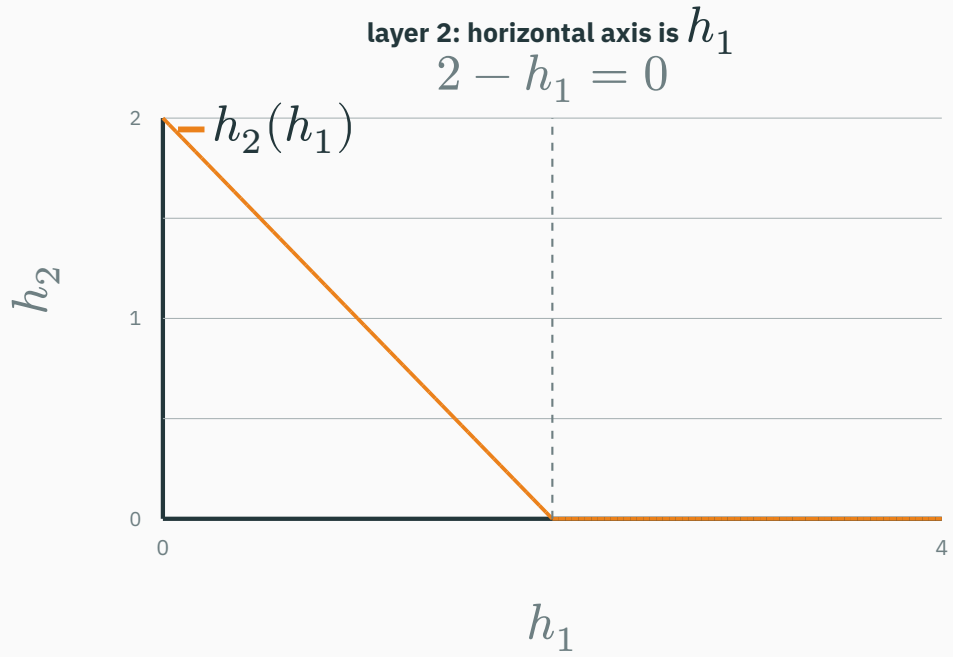

FIRST BEND

$$2x+1=0 \to x=-0.5$$

SECOND BEND IN INPUT SPACE

$$h_1=2 \to 2x+1=2 \to x=0.5$$

Composing the two maps gives three input regions

V VISUAL

$$h_2(x) = f_2(f_1(x)) = \mathrm{ReLU}(2 - \mathrm{ReLU}(2x + 1))$$

This illustrates sequential composition; it does not claim that this simple function requires depth.

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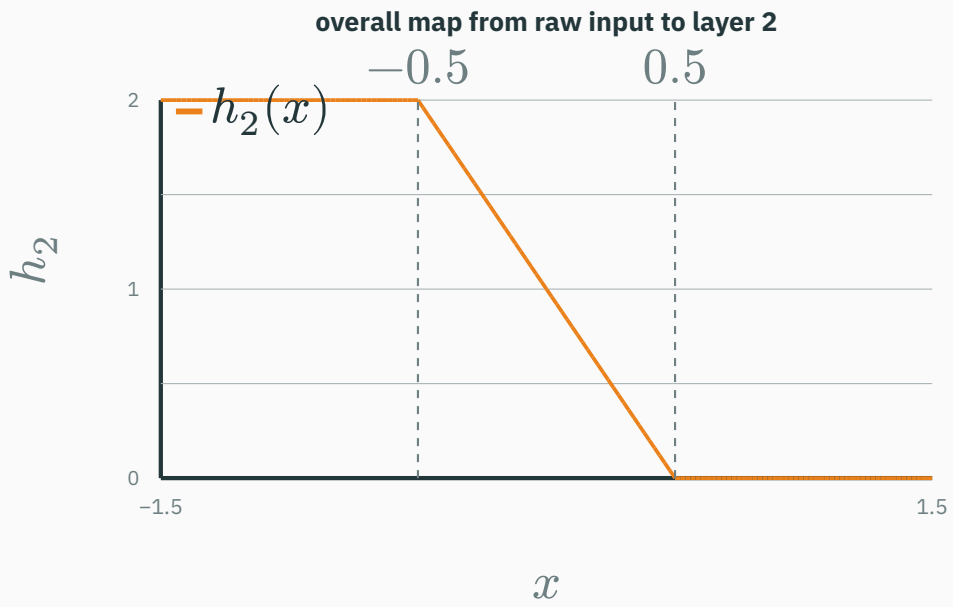


LEFT PLATEAU

$$x \le -0.5$$
$$h_1 = 0 \to h_2 = 2$$

MIDDLE RAMP

$$-0.5 < x < 0.5$$
$$h_1 = 2x + 1 \to h_2 = 1 - 2x$$

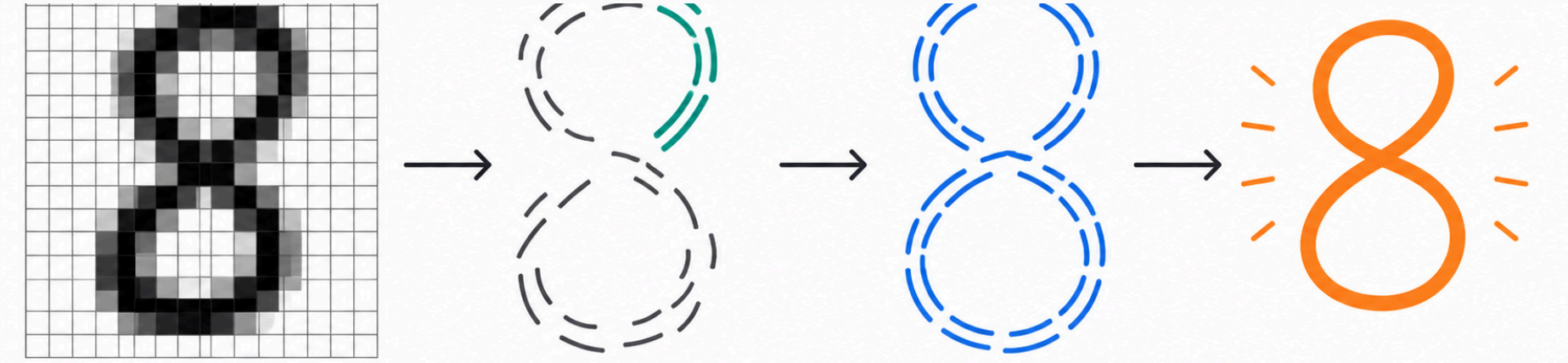
RIGHT PLATEAU

$$x \ge 0.5$$
$$h_1 \ge 2 \to h_2 = 0$$

Vision: depth builds strokes, then loops

V VISUAL

CONCEPTUAL SCHEMATIC · illustrates a possible hierarchy; not measured model activations



INPUT
pixel intensities

LAYER 1
curved stroke

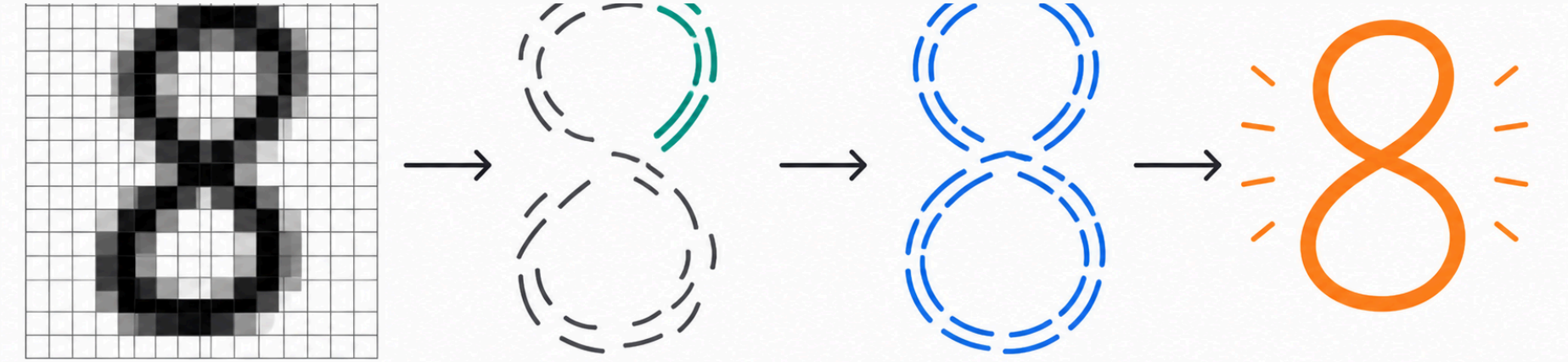
LAYER 2
two closed loops

OUTPUT
evidence for 8

Vision: depth builds strokes, then loops

V VISUAL

CONCEPTUAL SCHEMATIC · illustrates a possible hierarchy; not measured model activations



INPUT	LAYER 1	LAYER 2	OUTPUT
pixel intensities	curved stroke	two closed loops	evidence for 8

Early units can respond to local edges; later units can combine them into shapes that support the final class.

Choose width for variety; choose depth for composition

D DERIVATION

ADD WIDTH

- adds more features **in parallel**;
- helps when many patterns live at the same abstraction level;
- costs wider activations and weight matrices.

Example: add hinge locations to follow a detailed curve.

Choose width for variety; choose depth for composition

D DERIVATION

ADD WIDTH

- adds more features **in parallel**;
- helps when many patterns live at the same abstraction level;
- costs wider activations and weight matrices.

Example: add hinge locations to follow a detailed curve.

ADD DEPTH

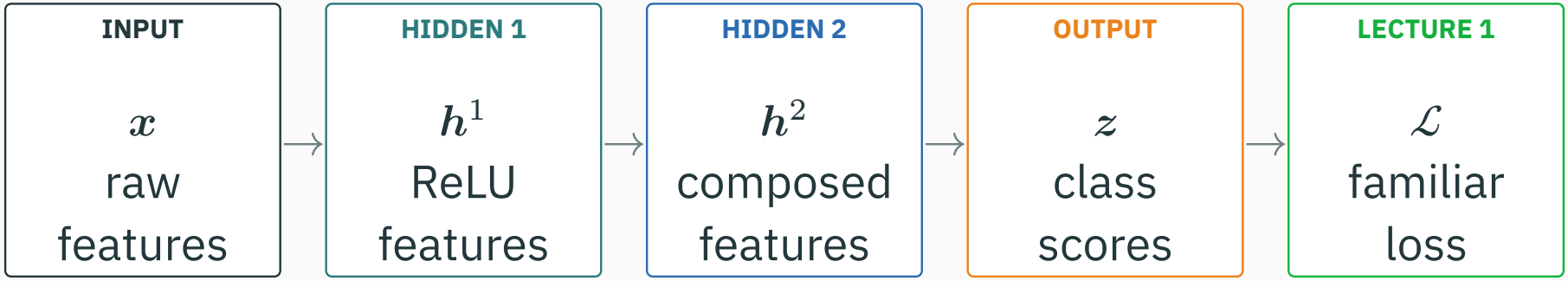
- adds more transformations **in sequence**;
- helps when the target is compositional or hierarchical;
- costs longer paths and harder optimization.

Example: strokes $\rightarrow$ loops $\rightarrow$ digit identity.

From representation to learning

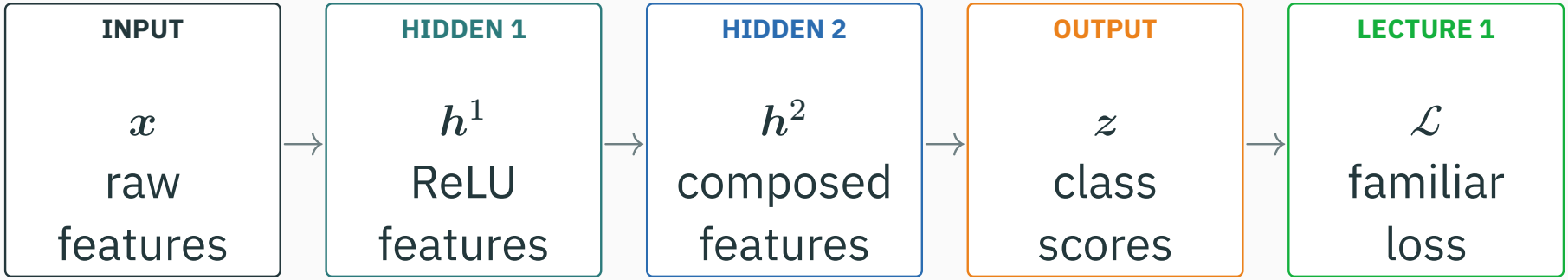
A richer representation; the same learning loop

V VISUAL



A richer representation; the same learning loop

V VISUAL



FORWARD

evaluate the current computation and scalar loss

NEXT · BACKPROP

compute how every parameter affected that loss

THEN · OPTIMIZE

use those gradients to change the parameters

We now have $x \to f_{\boldsymbol{\theta}}(x) \to \mathcal{L}$.

Next: how does the loss send credit backward through that computation?

computation graphs · local derivatives · backpropagation · automatic differentiation