## **Constrained Optimization I**

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# **Lagrangian and Duality**

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Lectures heavily inspired by the Maths for Machine learning book

• Minimax inequality states:  $\max_{\mathbf{y}} \min_{\mathbf{x}} q(\mathbf{x}, \mathbf{y}) \leqslant \min_{\mathbf{x}} \max_{\mathbf{y}} q(\mathbf{x}, \mathbf{y})$ 

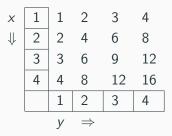
- Minimax inequality
  states:max<sub>y</sub> min<sub>x</sub> q(x, y) ≤ min<sub>x</sub> max<sub>y</sub> q(x, y)
- We first prove For all  $x, y = \min_{x} q(x, y) \leqslant \max_{y} q(x, y)$

• Let us choose q(x, y) = xy

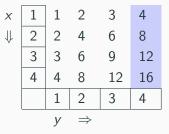
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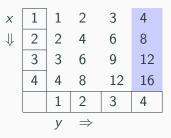


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- y = 4 maximizes  $q(x, y) \forall x$



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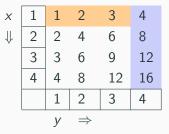
X	1	1	2	3	4
$\Downarrow$	2	2	4	6	8
	3	3	6	9	12
	4	4	8	12	16
		1	2	3	4
		У	$\Rightarrow$		

4

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- x = 1 minimizes  $q(x, y) \forall y$

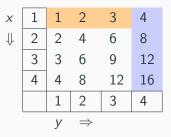
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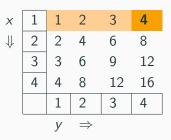


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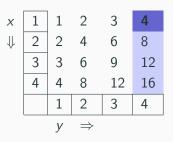
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- The equality occurs at x = 1, y = 4



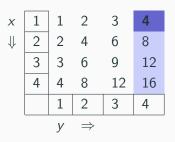
• Let us now find  $\max_{y} \min_{x} q(x, y)$ 



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- We can thus see our Minimax inequality  $\max_{\mathbf{y}} \min_{\mathbf{x}} q(\mathbf{x}, \mathbf{y}) \leqslant \min_{\mathbf{x}} \max_{\mathbf{y}} q(\mathbf{x}, \mathbf{y})$



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This would give infinte penalty if constraint is not satisfied. But, this formulation is hard to solve too.

Idea: Introduce Lagrange multipliers ( $\lambda_i \geq 0$ ) to "approximate" J(x)

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When  $\lambda \geqslant 0$ , the Lagrangian  $\mathcal{L}(x,\lambda)$  is a lower bound of J(x). Hence, the maximum of  $\mathfrak{L}(x,\lambda)$  with respect to  $\lambda$  is

$$J(\mathbf{x}) = \max_{\mathbf{\lambda} \geqslant 0} \mathfrak{L}(\mathbf{x}, \mathbf{\lambda})$$

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But, our original problem was minimizing J(x), which is equivalent to:

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We can write the dual objective as a function of  $\lambda$  as

$$\mathfrak{D}(\boldsymbol{\lambda}) = \min_{\boldsymbol{x} \in \mathbb{R}^d} \mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$$

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- For SVM like formulations, primal objective is the same as dual objective (strong duality)
- For some problems, there is a "daulity-gap" between the two objectives