Ridge Regression

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Introduction

A know measure of over-fitting can be the magnitude of the coefficient.

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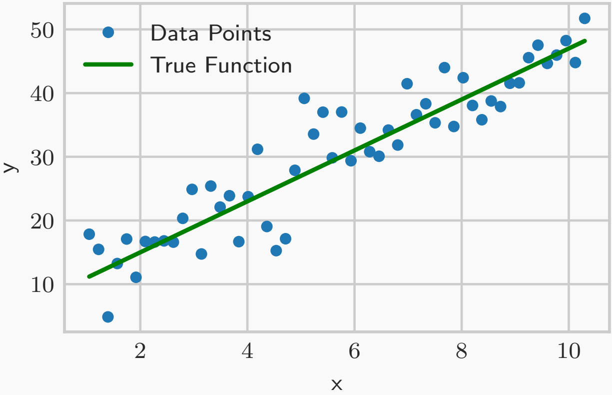
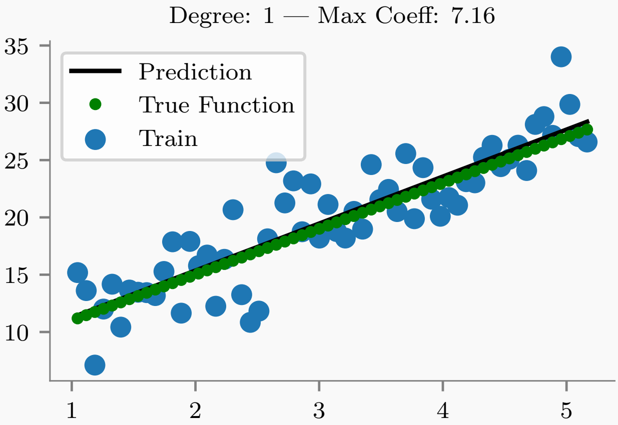
In $f(x) = c_0 + c_1 x + c_2 x^2 + \dots$ it is $\max |c_i|$

Introduction

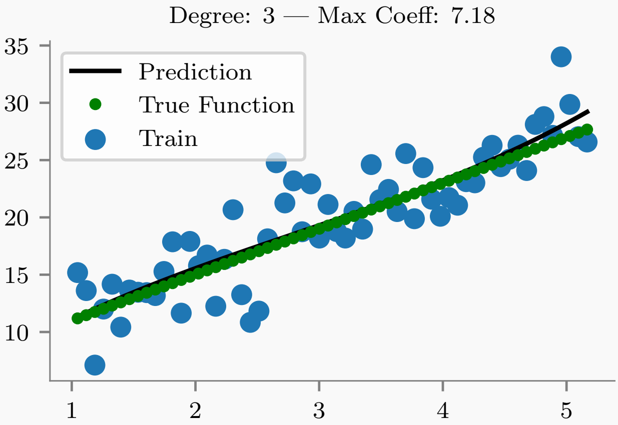


Base Data Set



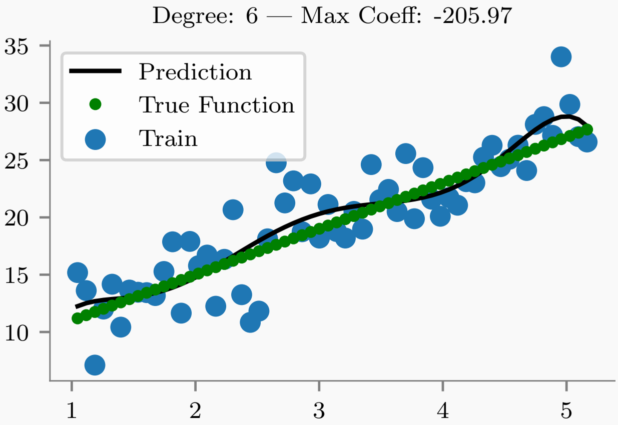
Fit with Degree 1

Introduction

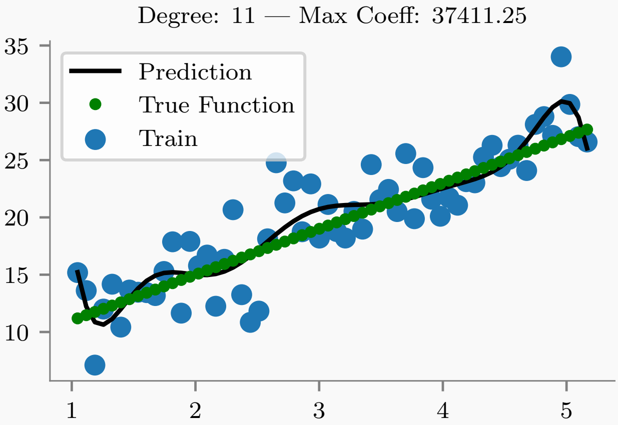


Fit with Degree 3

Introduction



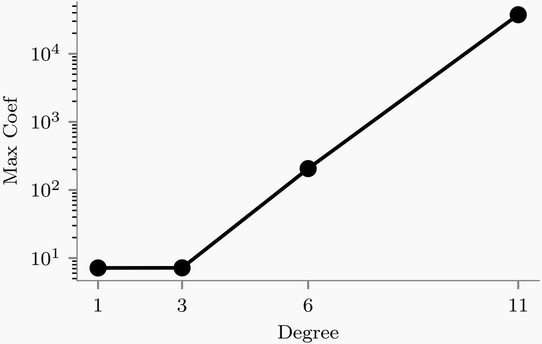
Fit with Degree 6



Fit with Degree 11

Introduction

In the examples we notice that as the degree increase (as the prediction starts to overfit the base data), the maximum coefficient also increases.



Trend of the coefficients

Introduction

To prevent over fitting we place penalties on large θ_i

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Objective

$$\text{Minimize } (y - X\theta)^T (y - X\theta)$$
$$\text{s.t. } \theta^T\theta \leq S$$

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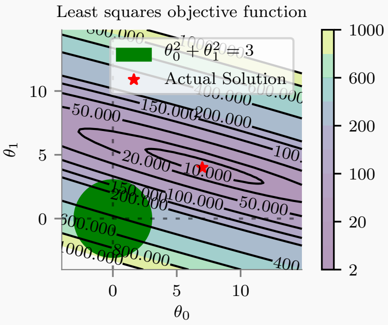
$$\text{s.t. } \theta^T\theta \leq S$$

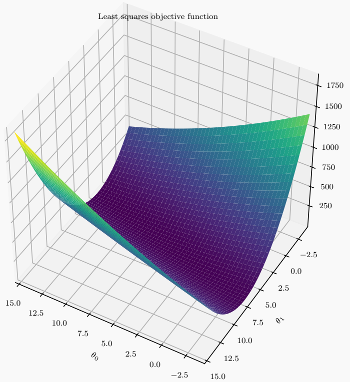
This is equivalent to

$$\text{Minimize } (y - X\theta)^T (y - X\theta) + \delta^2\theta^T\theta$$

Introduction



(a) Contour Plot



(b) Surface Plot

Visualization of the Example

KKT Conditions

To implement this we use KKT Conditions

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$$\text{Minimize } (y - X\theta)^T (y - X\theta)$$
$$\text{s.t. } \theta^T\theta \leq S$$
$$L(\theta, \mu) = (y - X\theta)^T (y - X\theta) + \mu \left(\theta^T\theta - S\right)$$

where, $\mu \geq 0$ (and $\mu = \delta^2$)

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If $\mu = 0$

There is no regularization

No effect on constraint

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If $\mu = 0$

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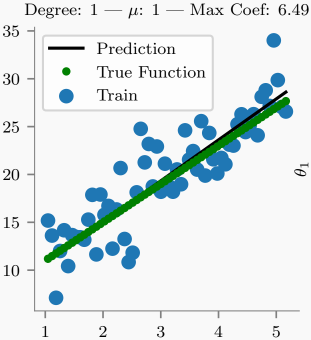
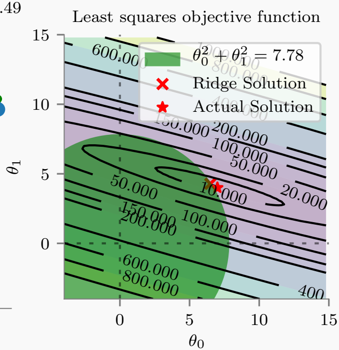
No effect on constraint

If $\mu \neq 0$

$\implies \theta^T\theta - S = 0$

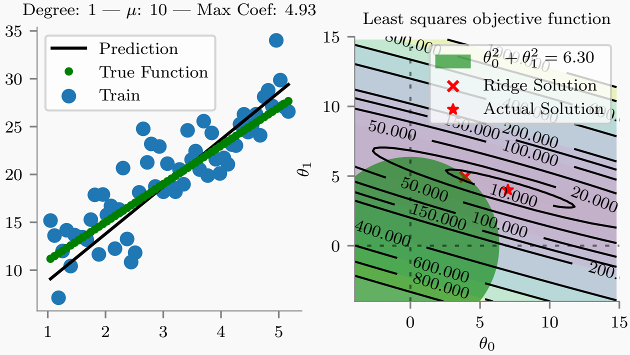
Effect of μ





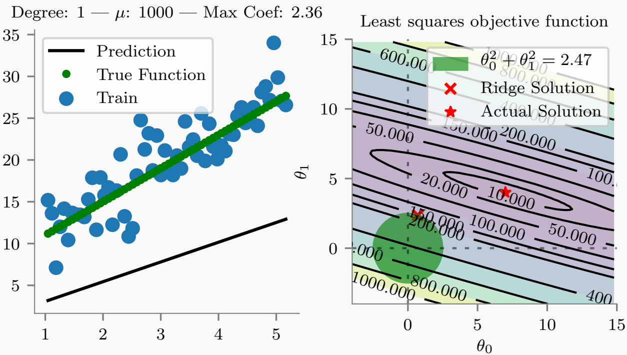
$$\mu = 1$$

Effect of μ



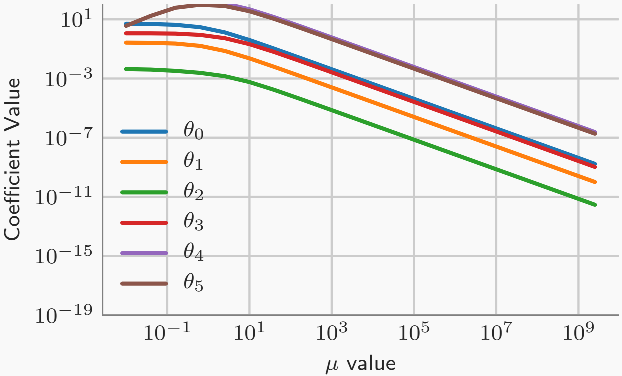
$\mu = 10$

Effect of μ



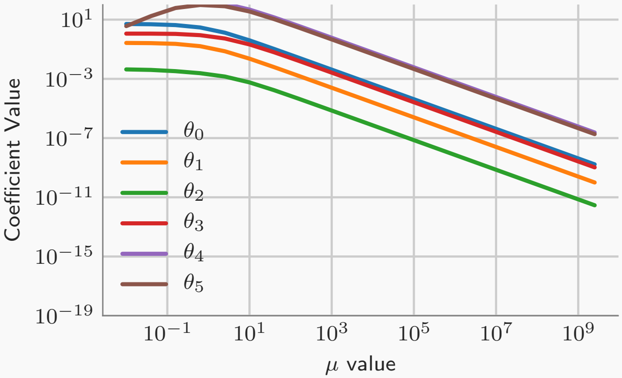
$\mu = 1000$

Effect of μ - Regularization of Parameters



Comparing the magnitudes of the coefficients with varying μ
(on the *Real Estate Data Set*)

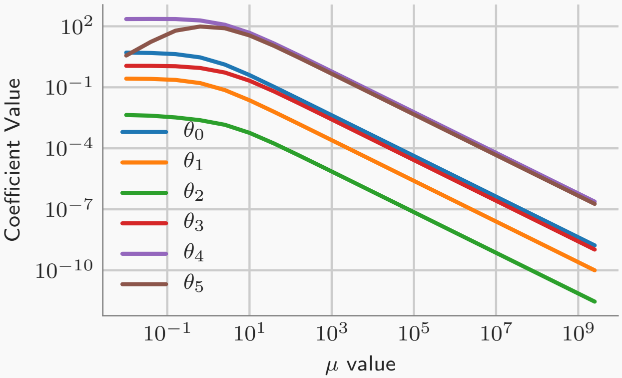
Effect of μ - Regularization of Parameters



Comparing the magnitudes of the coefficients with varying μ (on the *Real Estate Data Set*)

Are θ_i all zero for high μ?

Effect of μ - Regularization of Parameters



Comparing the magnitudes of the coefficients with varying μ (on the *Real Estate Data Set*)

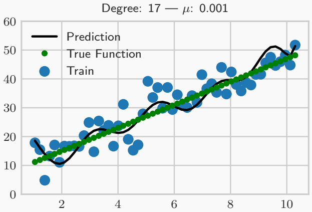
Analytical Method

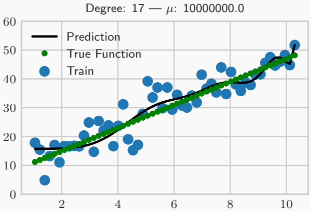
Ridge Objective

$$\min_{\theta} (y - X\theta)^T (y - X\theta) + \mu\theta^T\theta$$

$$\begin{aligned}
\frac{\partial L(\theta, \mu)}{\partial \theta} &= 0 \\
\frac{\partial}{\partial \theta}\left\{y^T y - 2y^T X\theta + \theta^T X^T X\theta\right\} + \frac{\partial}{\partial \theta}\mu\theta^T\theta &= 0 \\
\implies -X^T y + \left(X^T X + \mu I\right)\theta &= 0 \\
\implies \theta^* &= \left(X^T X + \mu I\right)^{-1} X^T y
\end{aligned}$$

Bias/Variance



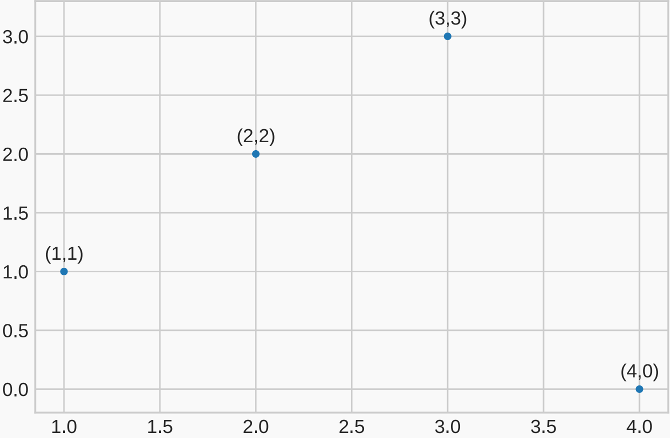


Fit High Order Polynomial
$\implies$ high variance
$\implies \mu \to 0$

Fit High Order Polynomial
$\implies$ low variance
$\implies \mu \to \infty$

Example

Q.) Solve Regularized ($\mu = 2$) and Unregularized.



Example: Unregularized

$$\theta = (X^T X)^{-1}(X^T y)$$

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$$\theta = (X^T X)^{-1}(X^T y)$$

$$X^T X = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$$

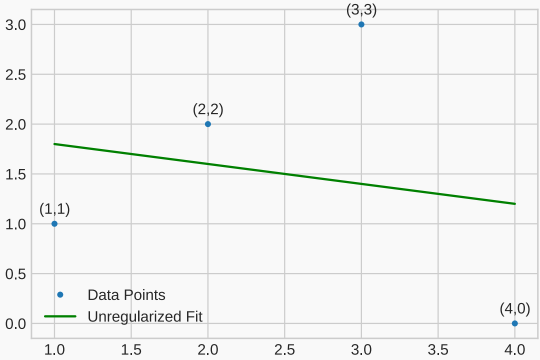
$$(X^T X)^{-1} = \frac{1}{20}\begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix}$$

$$X^T y = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

Example: Unregularized

$$\theta = (X^T X)^{-1}(X^T y)$$

$$\begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 2 \\ (-1/5) \end{bmatrix}$$



Example: Regularized

$$\theta = (X^T X + \mu I)^{-1}(X^T y)$$

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$$\theta = (X^T X + \mu I)^{-1}(X^T y)$$

$$X^T X = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$$

$$X^T X + \mu I = \begin{bmatrix} 6 & 10 \\ 10 & 32 \end{bmatrix}$$

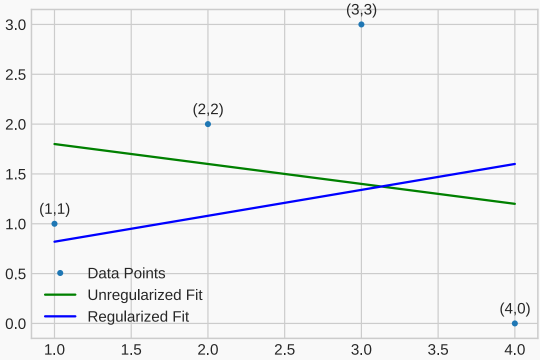
$$(X^T X + \mu I)^{-1} = \frac{1}{92}\begin{bmatrix} 32 & -10 \\ -10 & 6 \end{bmatrix}$$

$$X^T y = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

Example: Regularized

$$\theta = (X^T X + \mu I)^{-1}(X^T y)$$

$$\begin{bmatrix}\theta_0 \\ \theta_1\end{bmatrix} = \begin{bmatrix}0.56 \\ 0.26\end{bmatrix}$$



Multi-collinearity

$(X^T X)^{-1}$ is not computable when $|X^T X| = 0$.

This was a drawback of using linear regression

$$X = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & 3 & 6 \end{bmatrix}$$

The matrix X is not full rank.

Multi-collinearity

But with ridge regression, the matrix to be inverted is $X^T X + \mu I$ and not $X^T X$.

$$X^T X + \mu I = \begin{bmatrix} 3+\mu & 6 & 12 \\ 6 & 14+\mu & 28 \\ 12 & 28 & 56+\mu \end{bmatrix}$$

The matrix $X^T X$ would be full rank for $\mu > 0$.

Multi-collinearity

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Another interpretation of “regularisation”

Extension of the analytical model

For ridge with no penalty on θ_0

$$\hat{\theta} = \left(X^T X + \mu I^*\right)^{-1} X^T y$$

where,

$$I = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

TRUE FUNCTION: $y = x$

x	y
1	1
2	2

TRUE FUNCTION: $y = 100 + x$

x	y
1	101
2	102

TRUE FUNCTION: $y = x$

x_0	x	y
1	1	1
1	2	2

ADD COLUMN OF 1s

TRUE FUNCTION: $y = 100 + x$

x_0	x	y
1	1	101
1	2	102

TRUE FUNCTION: $y = x$

x_0	x	y
1	1	1
1	2	2

CASE I: $I = I_{2\times 2}$

$\mu = 100$

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1	1	101
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1	1	101
1	2	102

$$\hat{\theta} = (X^T X + \mu I)^{-1} X^T y$$

$$\hat{\theta} = [0.02 \quad 0.046]^T$$

TRUE FUNCTION: $y = x$

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1	1	1
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CASE I: $I = I_{2\times 2}$

$\mu = 100$

TRUE FUNCTION: $y = 100 + x$

x_0	x	y
1	1	101
1	2	102

$$\hat{\theta} = [1.9 \quad 2.8]^T$$

TRUE FUNCTION: $y = x$

x_0	x	y
1	1	1
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CASE I: $I = I_{2\times 2}$

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$$\hat{\theta} = (X^T X + \mu I)^{-1} X^T y$$

$$\hat{\theta} = [0.02 \quad 0.046]^T$$

$$\hat{y}(0) = 0.02$$

TRUE FUNCTION: $y = 100 + x$

x_0	x	y
1	1	101
1	2	102

$$\hat{\theta} = [1.9 \quad 2.8]^T$$

$$\hat{y}(0) = 1.9$$

TRUE FUNCTION: $y = x$

x_0	x	y
1	1	1
1	2	2

$$\hat{\theta} = (X^T X + \mu I^*)^{-1} X^T y$$

$$\hat{\theta} = [1.49 \ , 0.0049]^T$$

$$\hat{y}(0) = 1.49$$

CASE $\underline{\underline{2}}$: USE I^*

$\mu = 100$

TRUE FUNCTION: $y = 100 + x$

x_0	x	y
1	1	101
1	2	102

$$\hat{\theta} = [101 \ , \sim 0]^T$$

$$\hat{y}(0) = 101$$

TRUE FUNCTION: $y = x$

x_0	x	y
1	1	1
1	2	2

CASE $\underline{\underline{2}}$: USE I^*

$\mu = 100$

$$\hat{\theta} = (X^T X + \mu I^*)^{-1} X^T y$$

$$\hat{\theta} = [1.49 \; , 0.0049]^T$$

$$\hat{y}(0) = 1.49$$

$\Longrightarrow$

TENDS TOWARDS

$\bar{y}$

TRUE FUNCTION: $y = 100 + x$

x_0	x	y
1	1	101
1	2	102

$$\hat{\theta} = [101 \; , \sim 0]^T$$

$$\hat{y}(0) = 101$$

TENDS TOWARDS

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TENDS TOWARDS

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TRUE FUNCTION: $y = 100 + x$

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1	1	101
1	2	102

$$\hat{\theta} = [101 \, , \sim 0]^T$$

$$\hat{y}(0) = 101$$

TENDS TOWARDS

$\bar{y}$

ALTERNATIVE APPROACH

① TRANSFORM $y \rightarrow y'$ s.t. $\overline{y'} = 0$

$$y' = y - \overline{y}$$

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ALTERNATIVE APPROACH

1. TRANSFORM $y \rightarrow y'$ s.t. $\overline{y'} = 0$

$$y' = y - \overline{y}$$

2. TRAIN ON $\{(x_i, y'_i) \ \forall i\}$

3. PREDICT y' ON TEST $x_{\text{TEST}}(i)$ AND ADD $\bar{y}$ to get $\hat{y}$

ALTERNATIVE APPROACH

① TRANSFORM $y \rightarrow y'$ s.t. $\overline{y'} = 0$

$$y' = y - \overline{y}$$

② TRAIN ON $\{(x_i, y'_i) \ \forall i\}$

③ PREDICT y' ON TEST $x_{TEST}(i)$ AND ADD $\overline{y}$ to get $\hat{y}$

NO NEED TO USE I^* HERE

TRUE FUNCTION: $y = 100 + x$

x_0	x	y
1	1	101
1	2	102

$\bar{y} = 101.5$

TRUE FUNCTION: $y = 100 + x$

x_0	x	y	y'
1	1	101	-0.5
1	2	102	0.5

TRUE FUNCTION : $y = 100 + x$

x_0	x	y	y'
1	1	101	-0.5
1	2	102	0.5

$$\hat{\theta} = (X^T X + \mu I)^{-1} X^T y'$$

$$= [-0.0001, \quad 0.0047]^T$$

TRUE FUNCTION: $y = 100 + x$

x_0	x	y	y'
1	1	101	-0.5
1	2	102	0.5

$$\hat{\theta} = (X^T X + \mu I)^{-1} X^T y'$$

$$= [-0.0001, \quad 0.0047]^T$$

$$\hat{y}'(0) = 0$$

$$\hat{y}(0) = \hat{y}'(0) + \bar{y} = 101.5$$

RIDGE REGRESSION

WHAT 'μ' to use?

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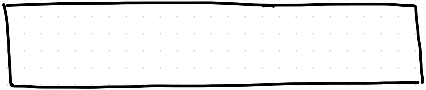
DATASET



RIDGE REGRESSION

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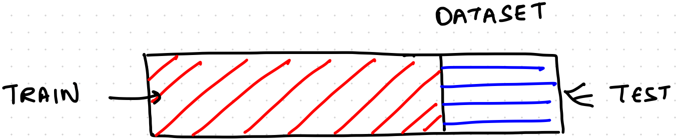
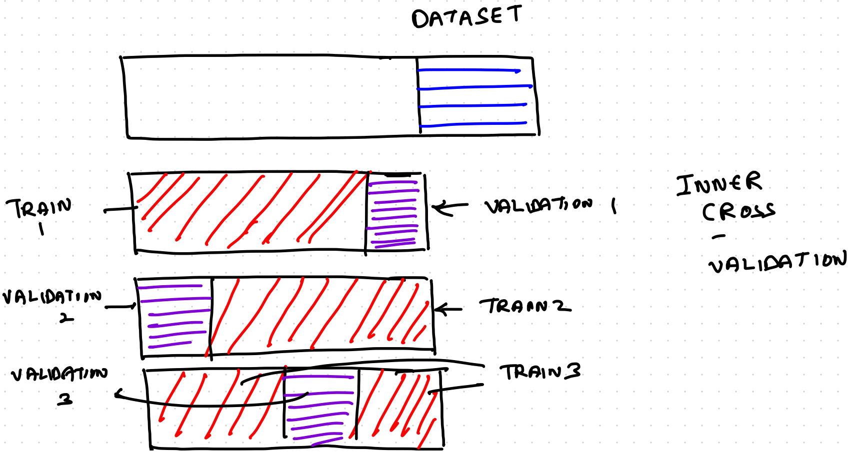


CROSS-VALIDATN (OUTER LOOP)

RIDGE REGRESSION

WHAT 'μ' to use?



RIDGE REGRESSION

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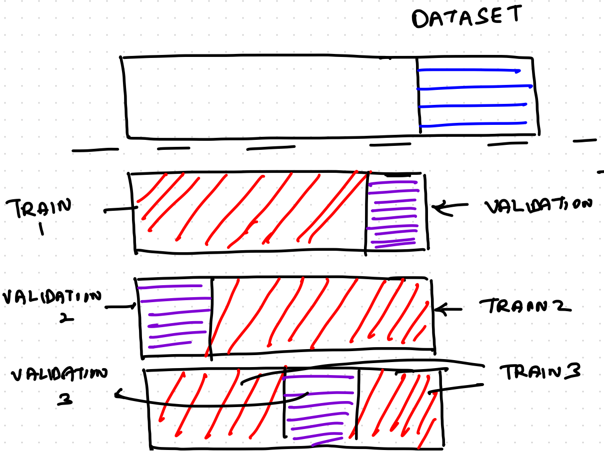
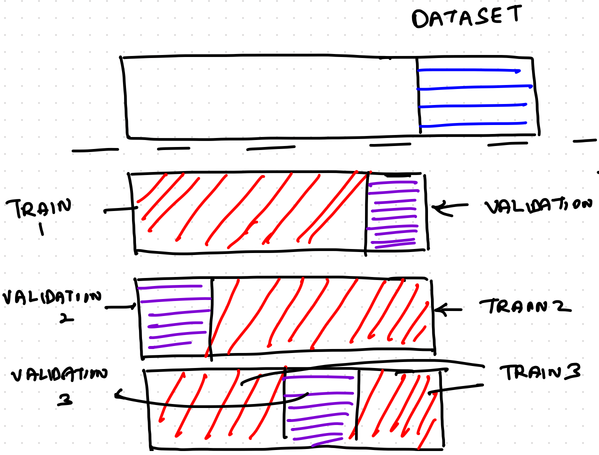


Typically 10^k

FOLD	MAE AT μ			
	10^{-4}	10^{-2}	1	10

RIDGE REGRESSION

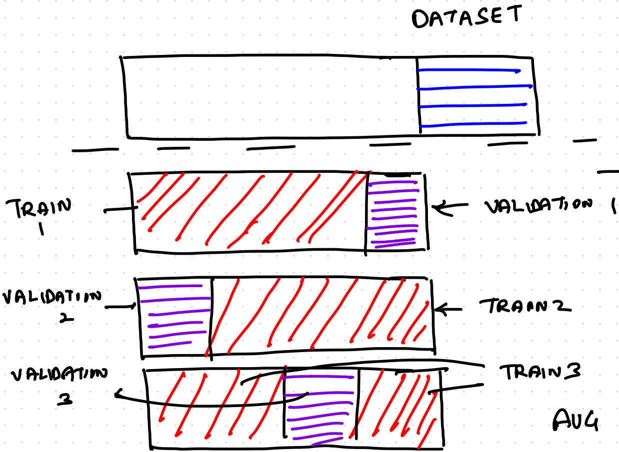
WHAT 'μ' to use?



FOLD	MAE AT μ			
	10^{-4}	10^{-2}	1	10
1	20	15	20	30
2	18	19	20	30
3	12	12	14	30

RIDGE REGRESSION

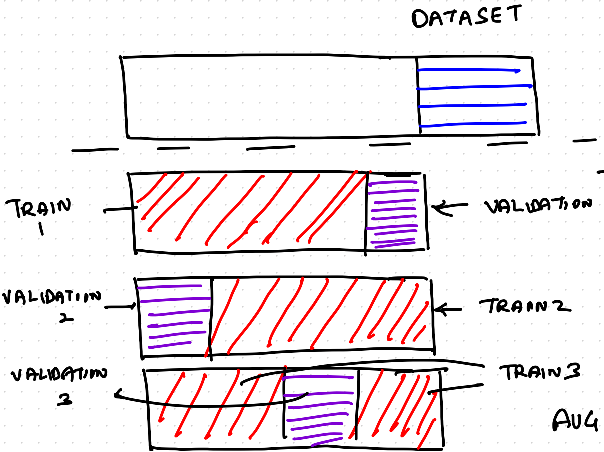
WHAT 'μ' to use?



FOLD	MAE AT μ			
	10^{-4}	10^{-2}	1	10
1	20	15	20	30
2	18	19	20	30
3	12	12	14	30
AVG	17	15	18	30

RIDGE REGRESSION

WHAT μ to use?

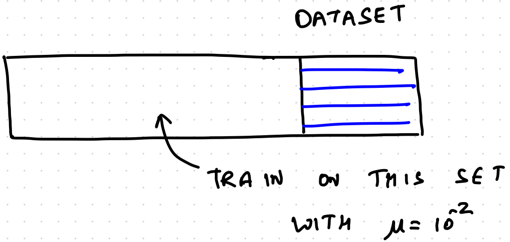


$\mu = 10^{-2}$ GIVES LOWEST VALIDATION ERROR

FOLD	MAE AT μ			
	10^{-4}	10^{-2}	1	10
1	20	15	20	30
2	18	19	20	30
3	12	12	14	30
AVG	17	15	18	30

RIDGE REGRESSION

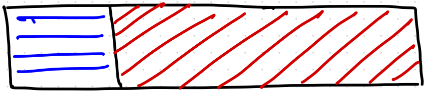
WHAT μ to use?



RIDGE REGRESSION

WHAT 'μ' to use?

DATASET



REPEAT PROCEDURE WITH OTHER 'OUTER LOOP' FOLDS

Ridge Solution using Gradient Descent

- $\theta = \theta - \alpha\frac{\partial}{\partial\theta}((y - X\theta)^\top (y - X\theta) + \mu\theta^\top\theta)$

Ridge Solution using Gradient Descent

- $\theta = \theta - \alpha \frac{\partial}{\partial \theta}((y - X\theta)^{\top}(y - X\theta) + \mu\theta^{\top}\theta)$
- $\theta = \theta - \alpha(-2X^{\top}y + 2X^{\top}X\theta + 2\mu I\theta)$

Ridge Solution using Gradient Descent

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- $\theta = \theta - \alpha(-2X^\top y + 2X^\top X\theta + 2\mu I\theta)$
- $\theta = (1 - 2\alpha\mu I)\theta - \alpha(-2X^\top y + 2X^\top X\theta)$

Ridge Solution using Gradient Descent

- $\theta = \theta - \alpha \frac{\partial}{\partial \theta}((y - X\theta)^\top (y - X\theta) + \mu \theta^\top \theta)$
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- $\theta = (1 - 2\alpha\mu I)\theta - \alpha(-2X^\top y + 2X^\top X\theta)$
- $\theta = \underbrace{(1 - 2\alpha\mu I)\theta}_{\text{Shrinking } \theta} - \alpha(-2X^\top y + 2X^\top X\theta)$

Ridge Solution using Gradient Descent

- $\theta = \theta - \alpha \frac{\partial}{\partial \theta}((y - X\theta)^\top (y - X\theta) + \mu \theta^\top \theta)$
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- $\theta = (1 - 2\alpha\mu I)\theta - \alpha(-2X^\top y + 2X^\top X\theta)$
- $\theta = \underbrace{(1 - 2\alpha\mu I)\theta}_{\text{Shrinking } \theta} - \alpha(-2X^\top y + 2X^\top X\theta)$

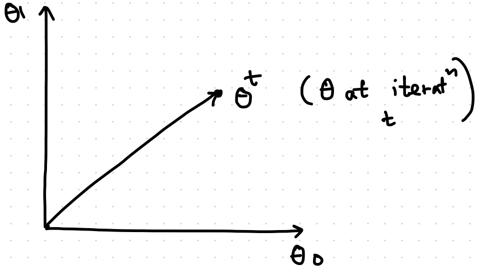
Ridge Solution using Gradient Descent

- $\theta = \theta - \alpha \frac{\partial}{\partial \theta}((y - X\theta)^\top (y - X\theta) + \mu\theta^\top\theta)$
- $\theta = \theta - \alpha(-2X^\top y + 2X^\top X\theta + 2\mu I\theta)$
- $\theta = (1 - 2\alpha\mu I)\theta - \alpha(-2X^\top y + 2X^\top X\theta)$
- $\theta = \underbrace{(1 - 2\alpha\mu I)\theta}_{\text{Shrinking } \theta} - \alpha(-2X^\top y + 2X^\top X\theta)$

- Contrast with update equation for unregularised regression:
- $\theta = \underbrace{\theta}_{\text{No Shrinking } \theta} - \alpha(-2X^\top y + 2X^\top X\theta)$

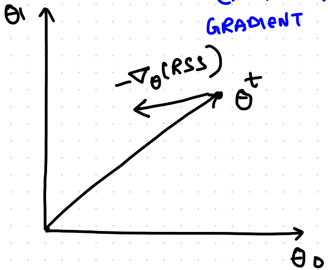
GD UPDATE FOR

UNREG. LINEAR REG.

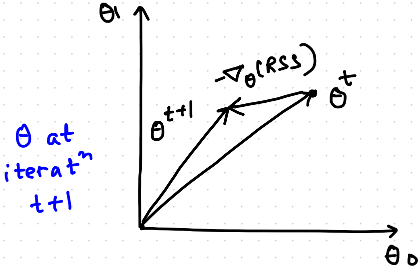


GD UPDATE FOR
UNREG. LINEAR REG.

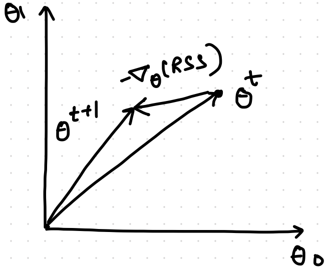
(NEGATIVE)
GRADIENT OF RSS WRT EVALUATED AT
COST θ θ^t



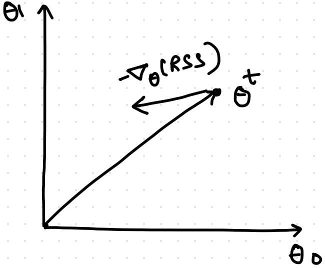
GD UPDATE FOR UNREG. LINEAR REG.



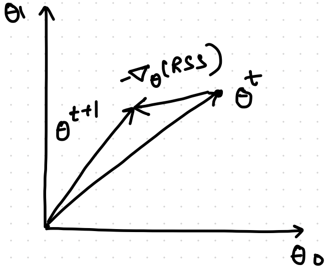
GD UPDATE FOR
UNREG. LINEAR REG.



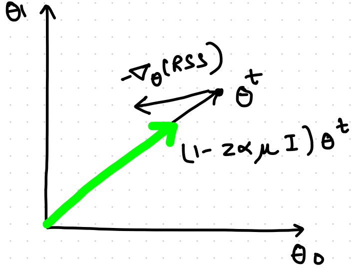
GD UPDATE FOR
RIDGE REGRESSION



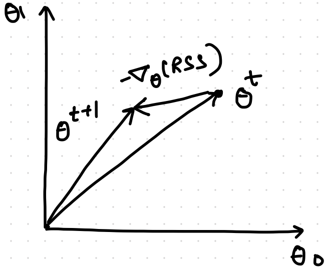
GD UPDATE FOR UNREG. LINEAR REG.



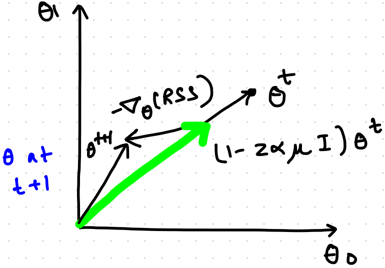
GD UPDATE FOR RIDGE REGRESSION



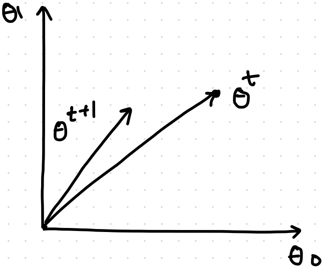
GD UPDATE FOR UNREG. LINEAR REG.



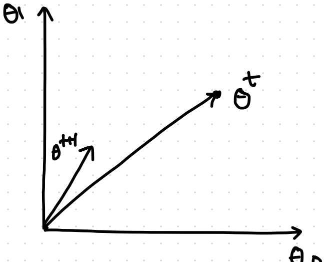
GD UPDATE FOR RIDGE REGRESSION



GD UPDATE FOR UNREG. LINEAR REG.



GD UPDATE FOR RIDGE REGRESSION



$\boxed{\text{Clearly, } \|\theta^{t+1}_{RIDGE}\|_2 \leq \|\theta^{t+1}_{UNREG}\|_2^2}$