Support Vector Machines

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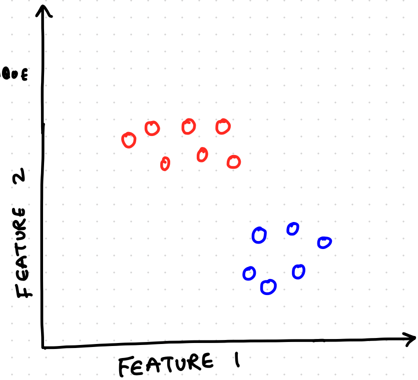
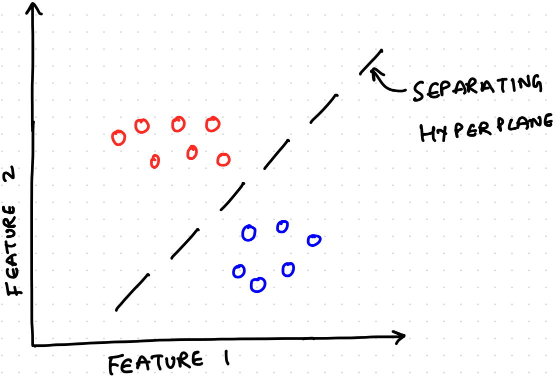
IIT Gandhinagar

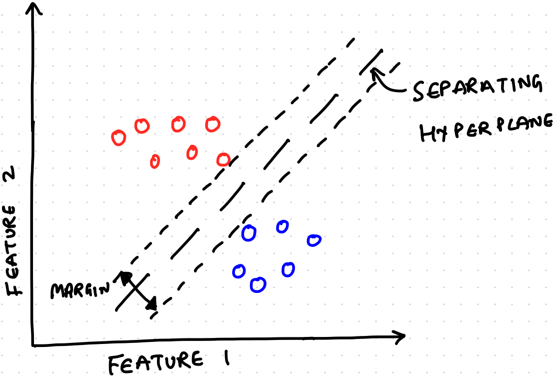
SUPPORT VECTOR MACHINES

POPULAR BINARY CLASSIFICATION TECHNIQUE

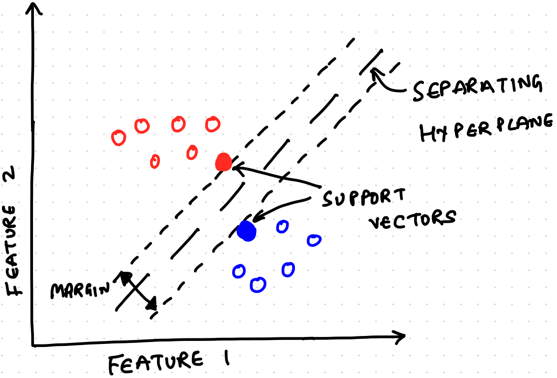




IDEA: DRAW A SEPARATING HYPERPLANE



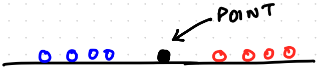
IDEA: MAXIMIZE THE MARGIN



SUPPORT VECTORS: POINTS ON BOUNDARY | MARGIN

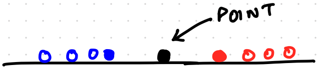
HYPERPLANE V/S # DIMENSIONS

1D

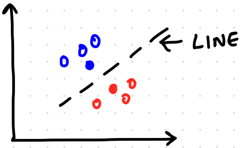


HYPERPLANE v/s # DIMENSIONS

1D

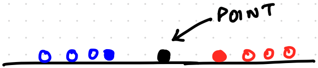


2D

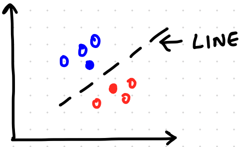


HYPERPLANE V/S # DIMENSIONS

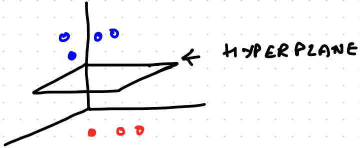
1D



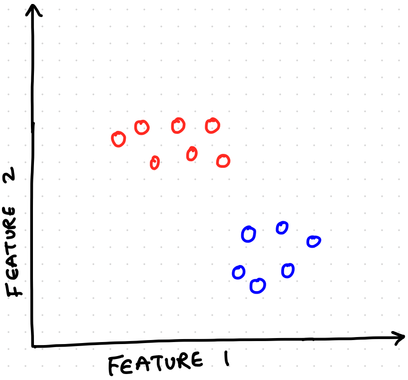
2D



3D (AND MORE)



WHICH HYPERPLANE?



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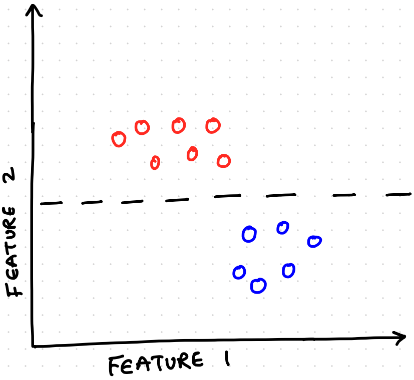

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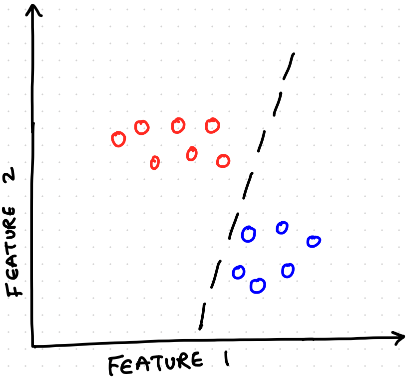

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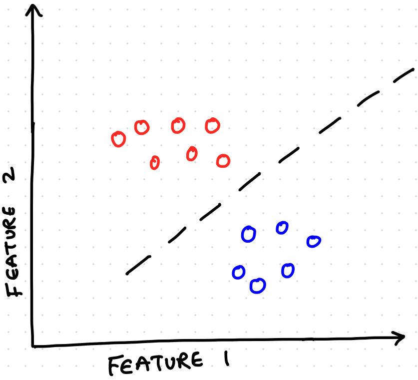
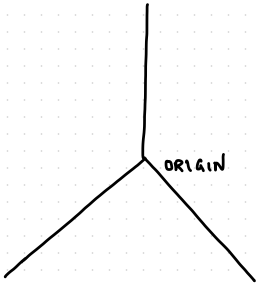
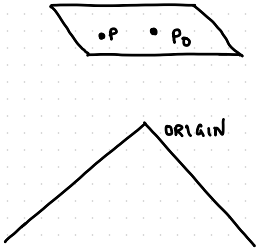


EQUATION OF HYPERPLANE

HOW TO DEFINE?



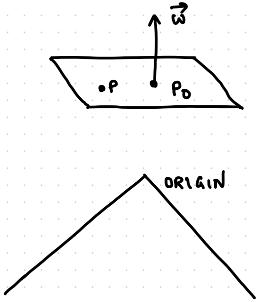
EQUATION OF HYPERPLANE



P: Any point on plane

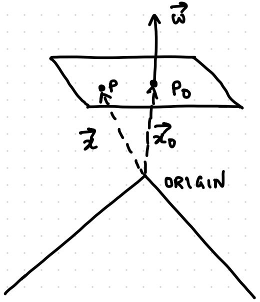
P_0: One point on plane

EQUATION OF HYPERPLANE



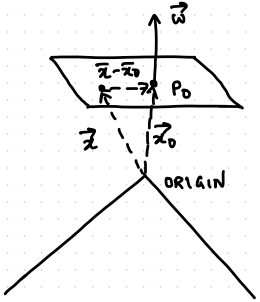
$\vec{w}$: $\perp$ vector to plane at P_0

EQUATION OF HYPERPLANE



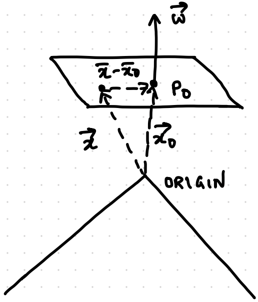
P and P_0 lie on plane

EQUATION OF HYPERPLANE



$\overrightarrow{P P_0} = \vec{x} - \vec{x}_0$ lies on plane

EQUATION OF HYPERPLANE



$\overrightarrow{P P_0} = \vec{x} - \vec{x}_0$ lies on plane

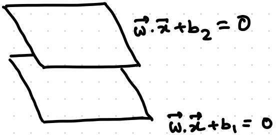
$\Rightarrow \vec{w} \perp (\vec{x} - \vec{x}_0)$

or, $\vec{w} \cdot (\vec{x} - \vec{x}_0) = 0$

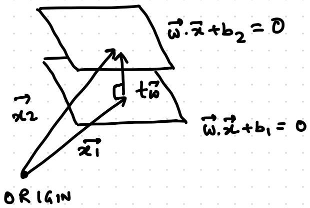
or, $\vec{w} \cdot \vec{x} - \vec{w} \cdot \vec{x}_0 = 0$

or, $\boxed{\vec{w} \cdot \vec{x} + b = 0}$

DISTANCE B/W || HYPERPLANES



DISTANCE B/W ∥ HYPERPLANES



Distance between 2 parallel hyperplanes

Equation of two planes is:

$$\vec{w} \cdot \vec{x} + b_1 = 0$$

$$\vec{w} \cdot \vec{x} + b_2 = 0$$

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For a point $\vec{x_1}$ on plane 1 and $\vec{x_2}$ on plane 2, we have:

Distance between 2 parallel hyperplanes

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For a point $\vec{x_1}$ on plane 1 and $\vec{x_2}$ on plane 2, we have:

$$\overrightarrow{x_2} = \overrightarrow{x_1} + t\vec{w}$$
$$D = |t\vec{w}| = |t|||\vec{w}||$$

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We can rewrite as follows:

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We can rewrite as follows:

$$\vec{w} \cdot \vec{x}_2 + b_2 = 0$$
$$\Rightarrow \vec{w} \cdot (\vec{x}_1 + t\vec{w}) + b_2 = 0$$

Distance between 2 parallel hyperplanes

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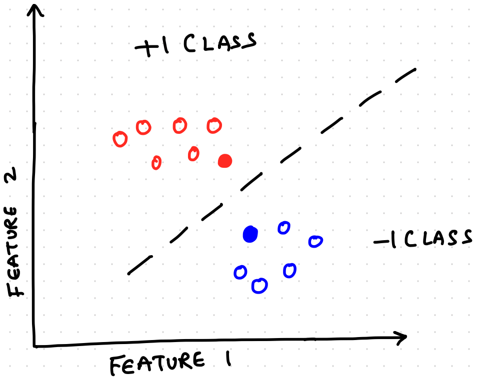
$$\Rightarrow \vec{w} \cdot (\vec{x}_1 + t\vec{w}) + b_2 = 0$$

$$\Rightarrow \vec{w} \cdot \vec{x}_1 + t\|\vec{w}\|^2 + b_1 - b_1 + b_2 = 0 \Rightarrow t = \frac{b_1 - b_2}{\|\vec{w}\|^2} \Rightarrow D = t\|\vec{w}\| = \frac{b_1 - b_2}{\|\vec{w}\|}$$

FORMULATION



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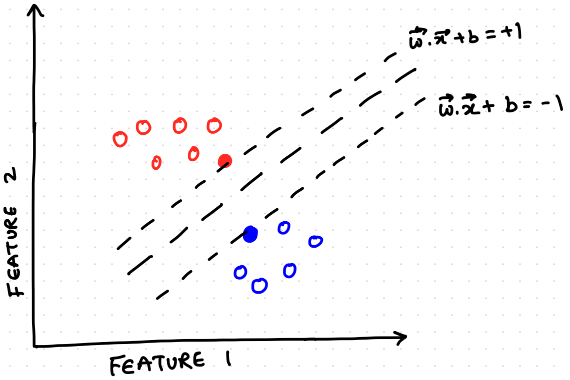
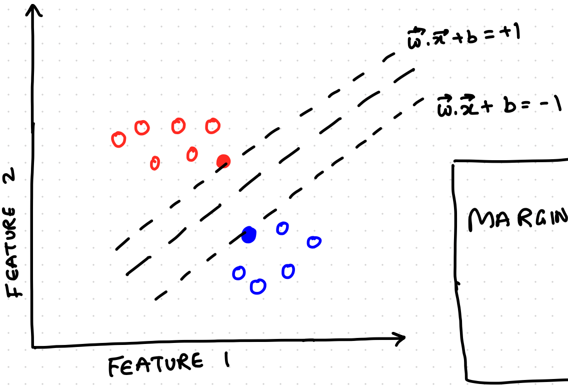


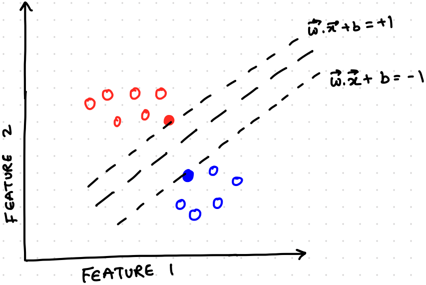
FORMULATION



$$\text{MARGIN} = \frac{(b+1)-(b-1)}{\|\vec{w}\|}$$

$$= \frac{2}{\|\vec{w}\|}$$

FORMULATION



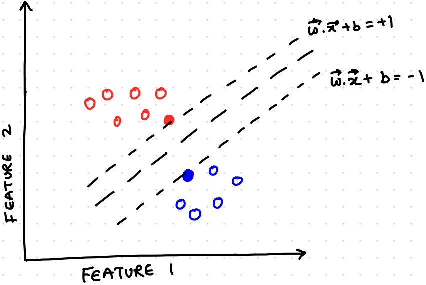
GOAL: MAXIMIZE MARGIN

$\Rightarrow$ MAXIMIZE $\dfrac{2}{\|\vec{w}\|}$

$\Rightarrow$ MINIMIZE $\|\vec{w}\|$

S.T. Correctly label points

FORMULATION



GOAL: MAXIMIZE MARGIN

$\Rightarrow$ MAXIMIZE $\dfrac{2}{||\vec{w}||}$

$\Rightarrow$ MINIMIZE $||\vec{w}||$

S.T. Correctly label points

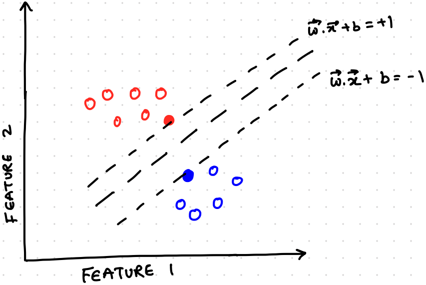
i.e. if $y_i = -1$

$\vec{w}.\vec{x}+b \leq -1$

if $y_i = +1$

$\vec{w}.\vec{x}+b \geq 1$

FORMULATION



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$$y_i(\vec{w}.\vec{x}+b) \geq 1$$

Primal Formulation

Objective

$$\text{Minimize } \frac{1}{2}||w||^2$$
$$\text{s.t. } y_i(w.x_i + b) \geq 1 \ \ \forall i$$

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Q) What is $||w||$?

Primal Formulation

Objective

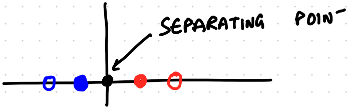
$$\text{Minimize } \frac{1}{2}||w||^2$$
$$\text{s.t. } y_i(w.x_i + b) \geq 1 \ \forall i$$

Q) What is $||w||$?

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix}$$

$$||w|| = \sqrt{w^T w}$$
$$= \sqrt{\begin{bmatrix} w_1, w_2, \dots w_n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix}}$$

EXAMPLE (IN 1D)



Simple Exercise

$$\begin{bmatrix} x & y \\ 1 & 1 \\ 2 & 1 \\ -1 & -1 \\ -2 & -1 \end{bmatrix}$$

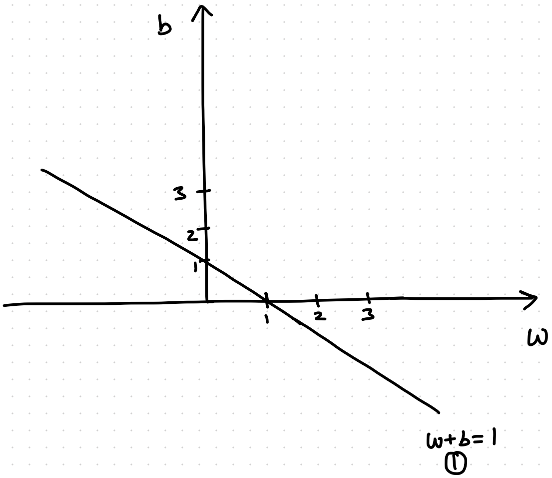
Separating Hyperplane: $wx + b = 0$

Simple Exercise

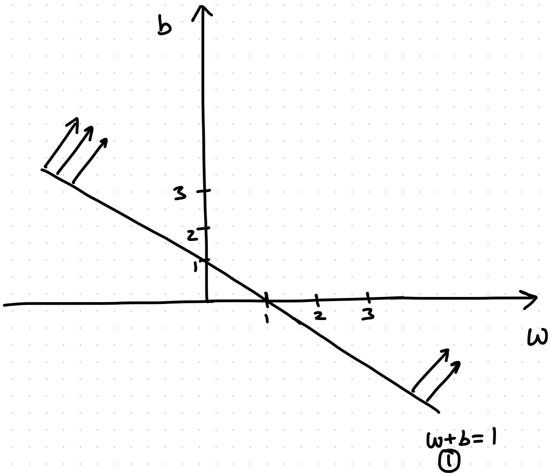
$$y_i(w_i x_i + b) \geq 1$$

$$\begin{bmatrix} x_1 & y \\ 1 & 1 \\ 2 & 1 \\ -1 & -1 \\ -2 & -1 \end{bmatrix}$$

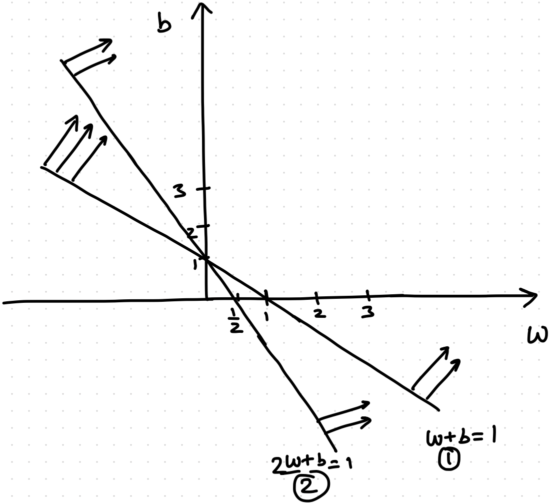
$$\Rightarrow y_i(w_i x_i + b) \geq 1$$
$$\Rightarrow 1(w_1 + b) \geq 1$$
$$\Rightarrow 1(2w_1 + b) \geq 1$$
$$\Rightarrow -1(-w_1 + b) \geq 1$$
$$\Rightarrow -1(-2w_1 + b) \geq 1$$

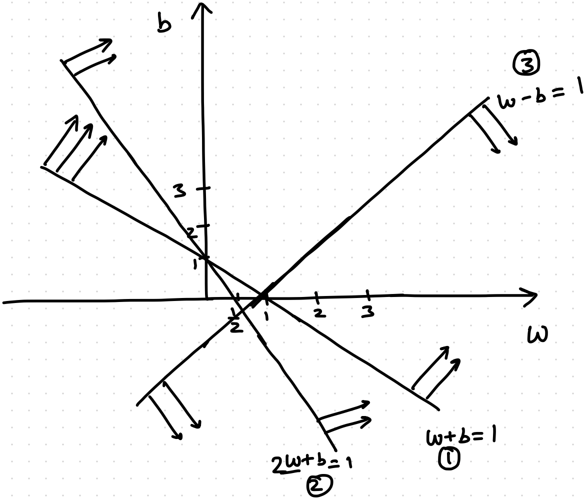


b
3
2
1
1
2
3
w
w+b=1
①

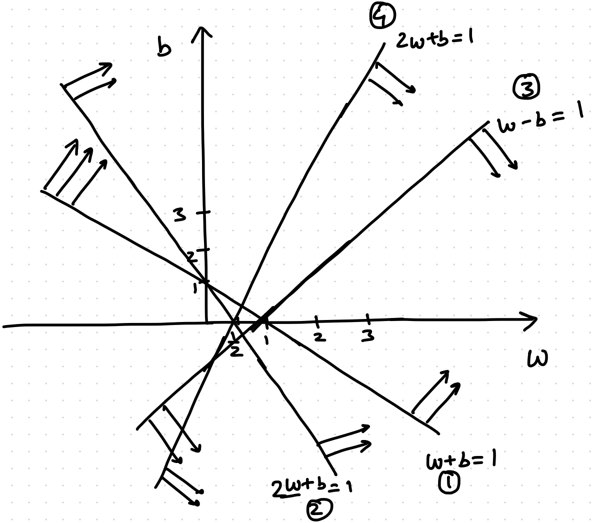


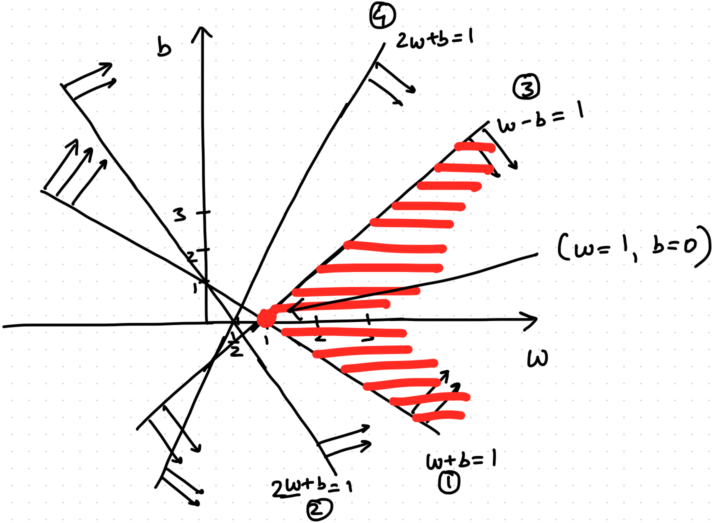
b
3
2
1
1
2
3
w
w+b=1
①


b
3
2
1
1/2
1
2
3
w
2w+b=1
②
w+b=1
①



b
w
3
2
1
1/2
1
2
3
③
w - b = 1
w + b = 1
①
2w + b = 1
②


b
w
3
2
1
1/2
1
2
3
④ 2w+b=1
③ w-b=1
w+b=1 ①
2w+b=1 ②



b
w
3
2
1
1/2
1
2
3
④ 2w+b=1
③ w-b=1
(w=1, b=0)
w+b=1 ①
2w+b=1 ②

Simple Exercise

$$w_{min} = 1, b = 0$$

$$w.x + b = 0$$

$$x = 0$$

Simple Exercise

Minimum values satisfying constraints $\Rightarrow w = 1$ and $b = 0$

$\therefore$ Max margin classifier $\Rightarrow x = 0$

Primal Formulation is a Quadratic Program

Generally;

$\Rightarrow$ Minimize Quadratic(x)

$\Rightarrow$ such that, Linear(x)

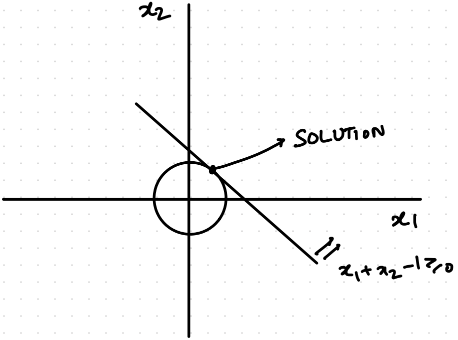
Question

$$x = (x_1, x_2)$$

$$\text{minimize} \quad \frac{1}{2}||x||^2$$

$$: x_1 + x_2 - 1 \geq 0$$

MINIMIZE QUADRATIC
s.t. LINEAR



Converting to Dual Problem

Primal $\Rightarrow$ Dual Conversion using Lagrangian multipliers

$$\text{Minimize } \frac{1}{2}||\bar{w}||^2$$
$$\text{s.t. } y_i(\bar{w}.x_i + b) \geq 1$$
$$\forall i$$

$$L(\bar{w}, b, \alpha_1, \alpha_2, ...\alpha_n) = \frac{1}{2}\sum_{i=1}^{d} w_i^2 - \sum_{i=1}^{N} \alpha_i(y_i(\bar{w}.\bar{x_i} + b) - 1) \;\; \forall \;\; \alpha_i \geq 0$$

$$\frac{\partial L}{\partial b} = 0 \Rightarrow \sum_{i=1}^{n} \alpha_i y_i = 0$$

Converting to Dual Problem

$$\frac{\partial L}{\partial w} = 0 \Rightarrow \bar{w} - \sum_{i=1}^{n} \alpha_i y_i \bar{x_i} = 0$$

$$\bar{w} = \sum_{i=1}^{N} \alpha_i y_i \bar{x_i}$$

$$L(\bar{w}, b, \alpha_1, \alpha_2, ...\alpha_n) = \frac{1}{2} \sum_{i=1}^{d} w_i^2 - \sum_{i=1}^{N} \alpha_i (y_i(\bar{w}.\bar{x_i} + b) - 1$$

$$= \frac{1}{2} ||\bar{w}||^2 - \sum_{i=1}^{N} \alpha_i y_i \vec{w}_. \bar{x_i} - \sum_{i=1}^{N} \alpha_i y_i b + \sum_{i=1}^{N} \alpha_i$$

$$= \sum_{=1}^{N} \alpha_i + \frac{\left(\sum_i \alpha_i y_i \bar{x_i}\right)\left(\sum_j \alpha_j y_j \bar{x_j}\right)}{2} - \sum_i \alpha_i y_i \left(\sum_j \alpha_j y_j \bar{x_j} \right) \bar{x_i}$$

Converting to Dual Problem

$$L(\alpha) = \sum_{i=1}^{N} \alpha_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j y_i y_j \bar{x}_i \cdot \bar{x}_j$$

Minimize $\|\bar{w}\|^2 \Rightarrow$	Maximize $L(\alpha)$
$s.t$	$s.t$
$y_i(\bar{w}, x_i + b) \geqslant 1$	$\sum_{i=1}^{N} \alpha_i y_i = 0 \; \forall \; \alpha_i \geq 0$

Question

Question:

$\alpha_i \left(y_i \left(\bar{w}, \bar{x}_i + b \right) - 1 \right) = 0 \quad \forall i$ as per KKT slackness

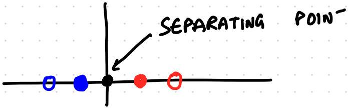
What is α_i for support vector points?

Answer: For support vectors,

$$\bar{w}.\bar{x}_i + b = -1 \text{ (+ve class)}$$
$$\bar{w}.\bar{x}_i + b = +1 \text{ (+ve class)}$$

$y_i \left(\bar{w} \cdot \bar{x}_i + b \right) - 1 \right) = 0 \quad$ for $i = \{$support vector points$\}$
$\therefore \alpha_i$ where i $\in \{$support vector points$\} \neq 0$
For all non-support vector points $\alpha_i = 0$

EXAMPLE (IN 1D)



Revisiting the Simple Example

$$\begin{bmatrix} x_1 & y \\ 1 & 1 \\ 2 & 1 \\ -1 & -1 \\ -2 & -1 \end{bmatrix}$$

$$L(\alpha) = \sum_{i=1}^{4} \alpha_i - \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} \alpha_i \alpha_j y_i y_j \bar{x_i} \bar{x_j} \qquad \alpha_i \geq 0$$

$$\sum \alpha_i y_i = 0 \qquad \alpha_i(y_i(\bar{w}.\bar{x_i} + b - 1) = 0$$

Revisiting the Simple Example

$$
\begin{aligned}
L(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\
& - \frac{1}{2}\{\alpha_1\alpha_1 \times (1 * 1) \times (1 * 1) \\
& \quad + \\
& \alpha_1\alpha_2 \times (1 * 1) \times (1 * 2) \\
& \quad + \\
& \alpha_1\alpha_3 \times (1 * -1) \times (1 * 1) \\
& \quad ... \\
& \alpha_4\alpha_4 \times (-1 * -1) \times (-2 * -2)\}
\end{aligned}
$$

How to Solve? $\Rightarrow$ Use the QP Solver!!

Revisiting the Simple Example

For the trivial example,

We know that only x $= \pm 1$ will take part in the constraint actively.

Thus, $\alpha_2, \alpha_4 = 0$

By symmetry, $\alpha_1 = \alpha_3 = \alpha$ (say)

& $\sum y_i \alpha_i = 0$

$$
\begin{aligned}
L(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = & 2\alpha \\
& - \frac{1}{2} \{ \alpha^2 (1)(-1)(1)(-1) \\
& \quad + \alpha^2 (-1)(1)(-1)(1) \\
& \quad + \alpha^2 (1)(1)(1)(1) + \alpha^2 (-1)(-1)(-1)(-1) \\
& \}
\end{aligned}
$$

$$\underset{\alpha}{Maximize} \quad 2\alpha - \tfrac{1}{2}(4\alpha^2)$$

Revisiting the Simple Example

$$\frac{\partial}{\partial \alpha}\left(2\alpha - 2\alpha^2\right) = 0 \Rightarrow 2 - 4\alpha = 0$$

$$\Rightarrow \ \alpha = 1/2$$

$$\therefore \alpha_1 = 1/2 \ \ \alpha_2 = 0; \ \ \alpha_3 = 1/2 \ \ \alpha_4 = 0$$

$$\vec{w} = \sum_{i=1}^{N} \alpha_i y_i \bar{x}_i = 1/2 \times 1 \times 1 + 0 \times 1 \times 2$$

$$+1/2 \times -1 \times -1 + 0 \times -1 \times -2$$

$$= 1/2 + 1/2 = 1$$

Finding b:

For the support vectors we have,

$y_i(\vec{w} \cdot \overrightarrow{x_i} + b) - 1 = 0$

or, $y_i\ (\bar{w} \cdot \bar{x}_1 + b) = 1$

or, $y_i^2\,(\bar{w} \cdot \bar{x}_i + b) = y_i$

or, $\quad \bar{w}, \bar{x}_i + b = y_i\ \ (\because y_i^2 = 1)$

or, $b = y_i - w \cdot x_i$

In practice, $b = \frac{1}{N_{SV}} \sum_{i=1}^{N_{SV}} (y_i - \bar{w}\bar{x}_i)$

Obtaining the Solution

$$
\begin{aligned}
b &= \frac{1}{2}\{(1-(1)(1)) + (-1-(1)(-1))\\
&= \frac{1}{2}\{0+0\} = 0\\
&= 0\\
&\therefore w = 1 \;\;\&\;\; b = 0
\end{aligned}
$$

Making Predictions

Making Predictions

$$\hat{y}(x_i) = \text{SIGN}(w \cdot x_i + b)$$

For $x_{test} = 3$; $\hat{y}(3) = \text{SIGN}(1 \times 3 + 0) = +$ve class

Making Predictions

Alternatively,

$$\hat{y}\left(x_{TEST}\right) = \text{SIGN}\left(\bar{w}\cdot\bar{x}_{TEST}+b\right)$$
$$= \text{SIGN}\left(\sum_{i=1}^{N_S}\alpha_j y_j x_j \cdot x_{test} + b\right)$$

In our example,

$\alpha_1 = 1/2; \alpha_2 = 0; \quad \alpha_3 = 1/2; \alpha_4 = 0$

$$\hat{y}(3) = \text{SIGN}\left(\frac{1}{2}\times 1\times(1\times 3)+0+\frac{1}{2}\times(-1)\times(-1\times 3)+0\right)$$
$$= \text{SIGN}\left(\frac{6}{2}\right) = \text{SIGN}(3) = +1$$

ORIGINAL DATA
IN R

Non-Linearly Separable Data

Non-Linearly Separable Data

Data not separable in $\mathbb{R}$

Non-Linearly Separable Data

Data not separable in $\mathbb{R}$
Can we still use SVM?

Non-Linearly Separable Data

Data not separable in $\mathbb{R}$

Can we still use SVM?

Yes!

Non-Linearly Separable Data

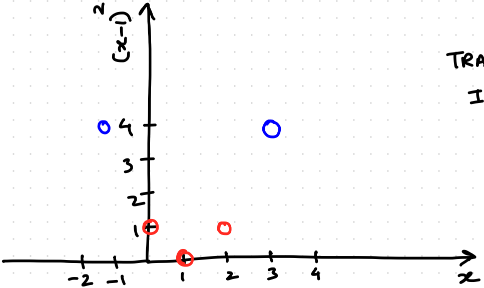
Data not separable in $\mathbb{R}$

Can we still use SVM?

Yes!

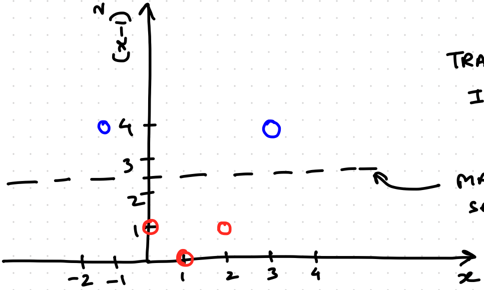
How? Project data to a higher dimensional space.

ORIGINAL DATA IN $\mathbb{R}$



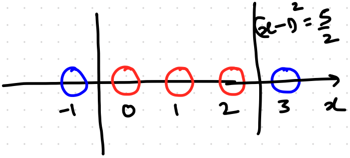
TRANSFORMED DATA IN $\mathbb{R}^2$

ORIGINAL DATA
IN R



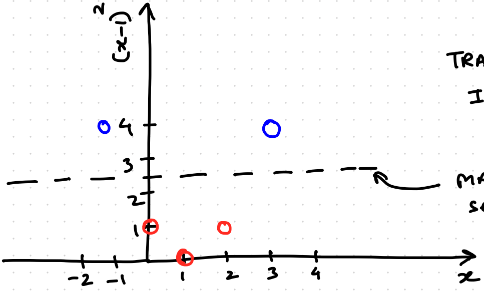
TRANSFORMED DATA
IN R^2

MAX MARGIN
SEPARATING
HYPERPLANE
$(x-1)^2 = \frac{5}{2}$



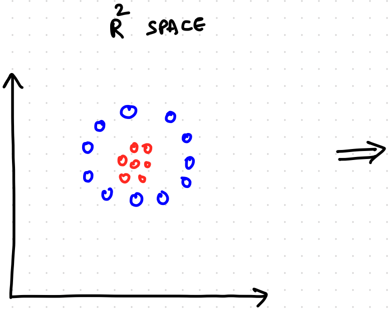
$(x-1)^2 = \frac{5}{2}$ [DECISION BOUNDARY IN R]

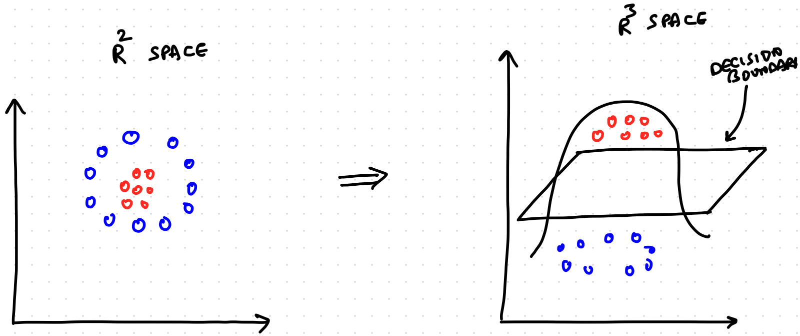
ORIGINAL DATA IN R



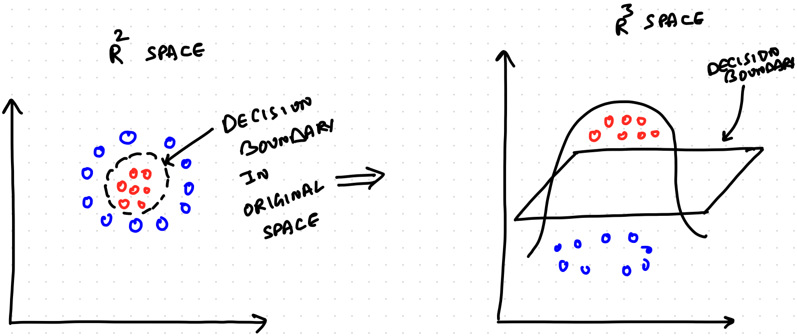
TRANSFORMED DATA IN R^2

MAX MARGIN SEPARATING HYPERPLANE $(x-1)^2 = \frac{5}{2}$


R^2 SPACE

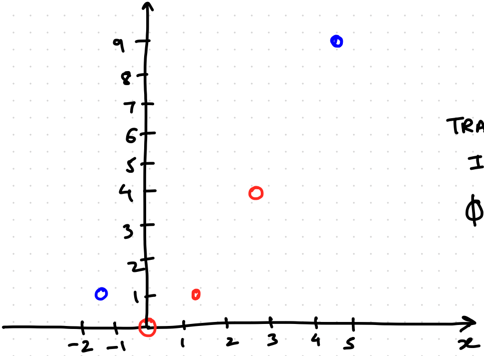


R^2 SPACE
R^3 SPACE
DECISION BOUNDARY



R^2 SPACE
DECISION BOUNDARY IN ORIGINAL SPACE
R^3 SPACE
DECISION BOUNDARY

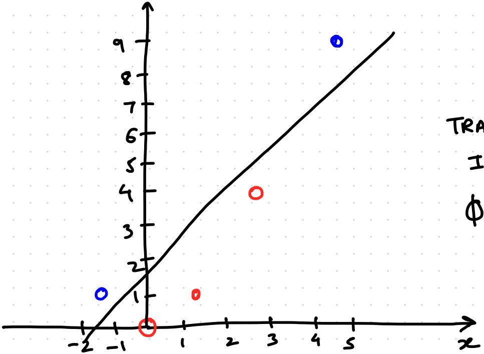
ORIGINAL DATA IN R



TRANSFORMED DATA IN R^2

$$\phi(x) = \begin{bmatrix} \sqrt{2}\, x \\ x^2 \end{bmatrix}$$

ORIGINAL DATA IN R



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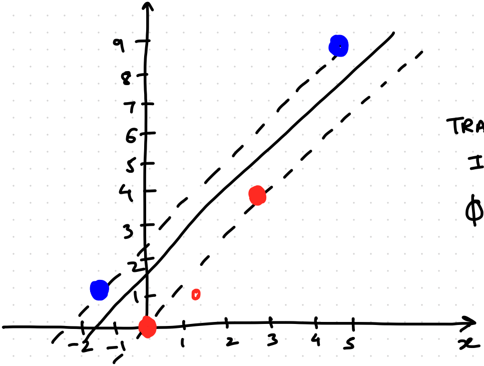
ORIGINAL DATA IN R



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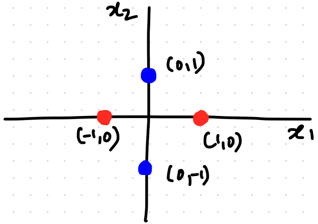
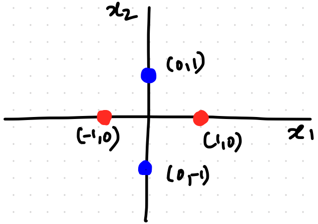
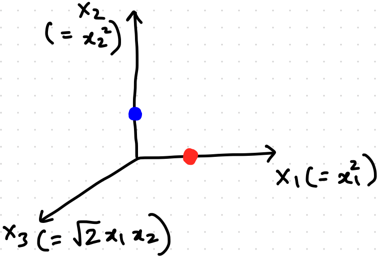


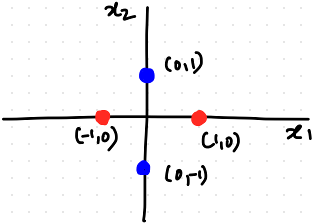
x_2
(0,1)
(-1,0)
(1,0)
x_1
(0,-1)



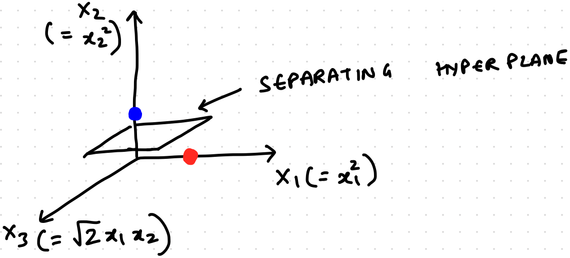
x2
(0,1)
(-1,0)
(1,0)
x1
(0,-1)



X2
(= x2²)
X1 (= x1²)
X3 (= √2 x1 x2)



x_2
(0,1)
(-1,0)
(1,0)
x_1
(0,-1)



X_2 $(= x_2^2)$
SEPARATING HYPERPLANE
$X_1 (= x_1^2)$
$X_3 (= \sqrt{2} x_1 x_2)$

Linear SVMs in higher dimensions

Linear SVM:

Maximize

$$L(\alpha) = \sum_{i=1}^{N} \alpha_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j y_i y_j \overline{x_i}.\overline{x_j}$$

such that constriants are satisfied.

$$\downarrow$$

Transformation (ϕ)

$$\downarrow$$

$$L(\alpha) = \sum_{i=1}^{N} \alpha_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j y_i y_j \phi(\overline{x_i}).\phi(\overline{x_j})$$

Linear SVMs in higher dimensions: Steps

1. Compute $\phi(x)$ for each point

$$\phi : \mathbb{R}^d \rightarrow \mathbb{R}^D$$

2. Compute dot products over $\mathbb{R}^D$ space

Q. If $D >> d$

Both steps are expensive!

Kernel Trick

Kernel Trick

- Can we compute $\mathrm{K}(\bar{x}_i, \bar{x}_j)$, such that

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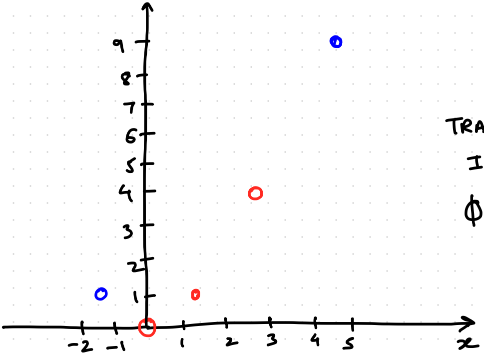
- $\mathsf{K}(\bar{x}_i, \bar{x}_j)$ is some function of dot product in original dimension

Kernel Trick

- Can we compute $K(\bar{x}_i, \bar{x}_j)$, such that
- $K(\bar{x}_i, \bar{x}_j) = \phi(\bar{x}_i).\phi(\bar{x}_j)$, where
- $K(\bar{x}_i, \bar{x}_j)$ is some function of dot product in original dimension
- $\phi(\bar{x}_i).\phi(\bar{x}_j)$ is dot product in high dimensions (after transformation)

KERNEL TRICK

ORIGINAL DATA IN $\mathbb{R}$



TRANSFORMED DATA IN $\mathbb{R}^2$

$$\phi(x) = \begin{bmatrix} \sqrt{2}\,x \\ x^2 \end{bmatrix}$$

KERNEL TRICK

$$\phi(x) = \begin{bmatrix} \sqrt{2}\,x \\ x^2 \end{bmatrix}$$

$$K(x_i, x_j) = \ ?$$

KERNEL TRICK

$$\phi(x) = \begin{bmatrix} \sqrt{2}\, x \\ x^2 \end{bmatrix}$$

$$K(x_i, x_j) = (1 + x_i \cdot x_j)^2 - 1$$

KERNEL TRICK

$$\phi(x) = \begin{bmatrix} \sqrt{2}\,x \\ x^2 \end{bmatrix}$$

$$K(x_i, x_j) = (1 + x_i \cdot x_j)^2 - 1$$

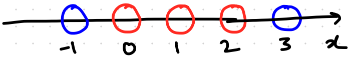
$$(1 + x_i \cdot x_j)^2 - 1 = 1 + 2x_i \cdot x_j + x_i^2 x_j^2 - 1$$

$$= 2x_i \cdot x_j + x_i^2 x_j^2$$

$$= (\sqrt{2}x_i \cdot \sqrt{2}x_j + x_i^2 \cdot x_j^2)$$

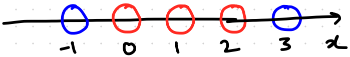
$$= \langle \sqrt{2}x_i, x_i^2 \rangle \cdot \langle \sqrt{2}x_j, x_j^2 \rangle$$

$$= \phi(x_i) \cdot \phi(x_j)$$



ORIGINAL DATASET

x	y
-1	-1
0	1
1	1
2	1
3	-1

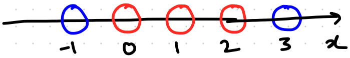


ORIGINAL DATASET

x	y
-1	-1
0	1
1	1
2	1
3	-1

TRANSFORMED DATASET

x	$\sqrt{2}x$	x^2	y
-1	$-\sqrt{2}$	1	-1
0	0	0	1
1	$\sqrt{2}$	1	1
2	$2\sqrt{2}$	4	1
3	$3\sqrt{2}$	9	-1



ORIGINAL DATASET

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0	1
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TRANSFORMED DATASET

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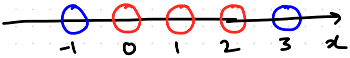
Calculation w/o Kernel Trick

$\phi(x_1) = \langle \sqrt{2}x, x^2 \rangle$: 2

$\phi(x_2) = \langle \sqrt{2}x, x^2 \rangle$: 2

$\phi(x_1) \cdot \phi(x_2) =$ 2 MULTIPLICATION + 1 Addition

} 7



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Calculation with Kernel Trick

$$K(x_1, x_2) = (1 + x_1 \cdot x_2)^2 - 1$$

$x_1 \cdot x_2 \rightarrow 1$

$1 + x_1 \cdot x_2 \rightarrow 1$

$(1 + x_1 \cdot x_2)^2 \rightarrow 1$

$(1 + x_1 \cdot x_2)^2 - 1 \rightarrow 1$

} 4

Kernel Trick

Q) Why did we use dual form?

Kernel Trick

Q) Why did we use dual form?

Kernels again!!

Kernel Trick

Q) Why did we use dual form?

Kernels again!!

Primal form doesn't allow for the kernel trick $\mathrm{K}\left(\bar{x}_1, \bar{x}_2\right)$ in dual and compute $\phi(x)$ and then dot product in D dimensions

Some Kernels

1. Linear: $\mathrm{K}(\bar{x}_1, \bar{x}_2) = \bar{x}_1 \bar{x}_2$
2. Polynomial: $\mathrm{K}(\bar{x}_1, \bar{x}_2) = (p + \bar{x}_1 \bar{x}_2)^q$
3. Gaussian: $\mathrm{K}(\bar{x}_1, \bar{x}_2) = e^{-\gamma||\bar{x}_1 - \bar{x}_2||^2}$ where $\gamma = \frac{1}{2\sigma^2}$ - Also called Radial Basis Function (RBF)

Kernels

Q) For $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ what space does kernel $\mathrm{K}(\bar{x}, \bar{x'}) = (1 + \bar{x}\bar{x'})^3$ belong to?

$\bar{x} \in \mathbb{R}^2$

$\phi(\bar{x}) \in \mathbb{R}^?$

$$\begin{aligned} K(x, z) &= (1 + x_1 z_1 + x_2 z_2)^3 \\ &= \ldots \\ &= < 1, x_1, x_2, x_1^2, x_2^2, x_1^2 x_2, x_1 x_2^2, x_1^3, x_2^3, x_1 x_2 > \end{aligned}$$

10 dimensional?

Does RBF involve dot product in lower-dimensional space?

Assuming x is a one-dimensional vector, we can rewrite the RBF kernel as:

$$K(x, z) = e^{-\gamma\|x-z\|^2} = e^{-\gamma(x-z)^2}$$

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Notice that the term $e^{2\gamma xz}$ is a dot product of the original data points x and z in the one-dimensional feature space.

What space does the RBF kernel lie in?

Q) For $\bar{x} = x$; what space does RBF kernel lie in?

$$\begin{aligned} K(x, z) &= e^{-\gamma ||x-z||^2} \\ &= e^{-\gamma (x-z)^2} \end{aligned}$$

Now:

$$e^{\alpha} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}$$

$\therefore e^{-\gamma (x-z)^2}$ is ∞ dimensional!!

Interpretation of RBF

- $\hat{y}(x) = sign(\sum_{i=1}^{n} \alpha_i y_i K(x, x_i) + b)$

Interpretation of RBF

- $\hat{y}(x) = sign(\sum_{i=1}^{n} \alpha_i y_i K(x, x_i) + b)$
- $K(x, x_i) = e^{-\gamma ||x - x_i||^2}$ is the RBF kernel evaluated at x and x_i

Interpretation of RBF

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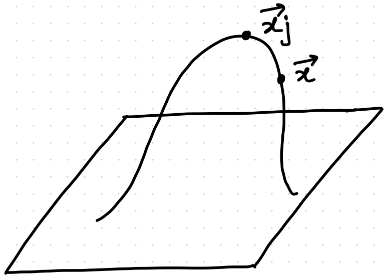
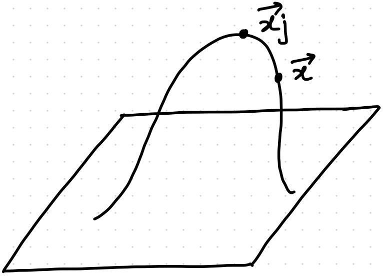
- $\hat{y}(x) = sign(\sum \alpha_i y_i e^{-\gamma||x-x_i||^2} + b)$
- $-||x - x_i||^2$ corresponds to radial term
- $\sum \alpha_i y_i$ is the activation component
- $e^{-||x-x_i||^2}$ is the basis component

RBF INTERPRETATION

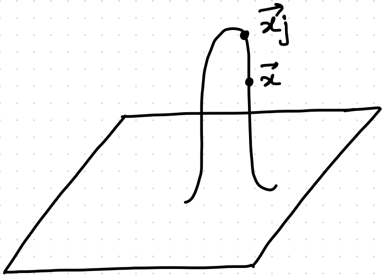


RBF INTERPRETATION



LOW γ

HIGH INFLUENCE OF $\vec{x}_j$



HIGH γ

LOW INFLUENCE OF $\vec{x}_j$