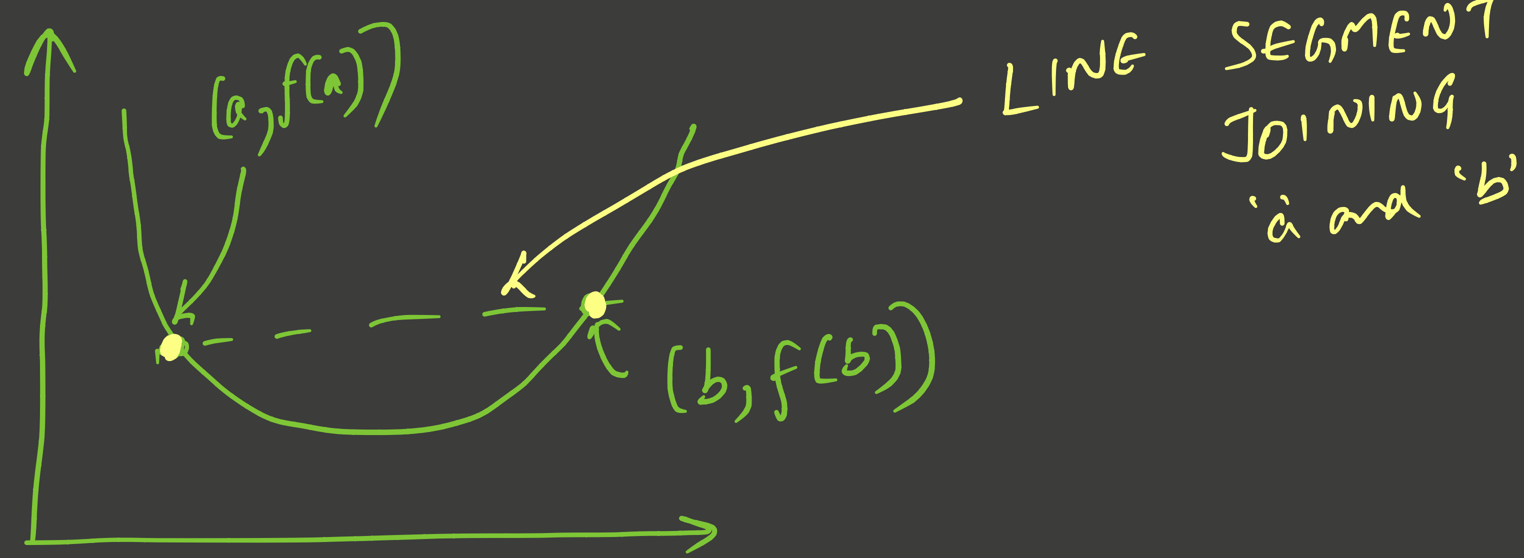
CONVEX FUNCTIONS

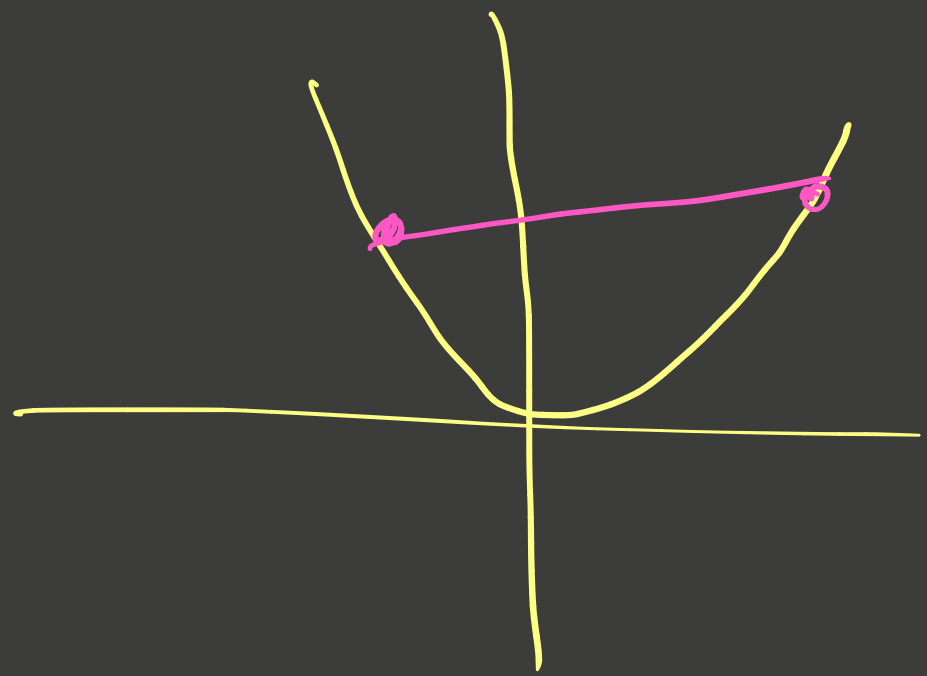
* DEFINED ON INTERVAL (a, b)



DEFINITION

LINE SEGMENT JOINING $(a, f(a))$ and $(b, f(b))$ SHOULD BE ABOVE OR ON THE FUNCTION

$y = x^2$

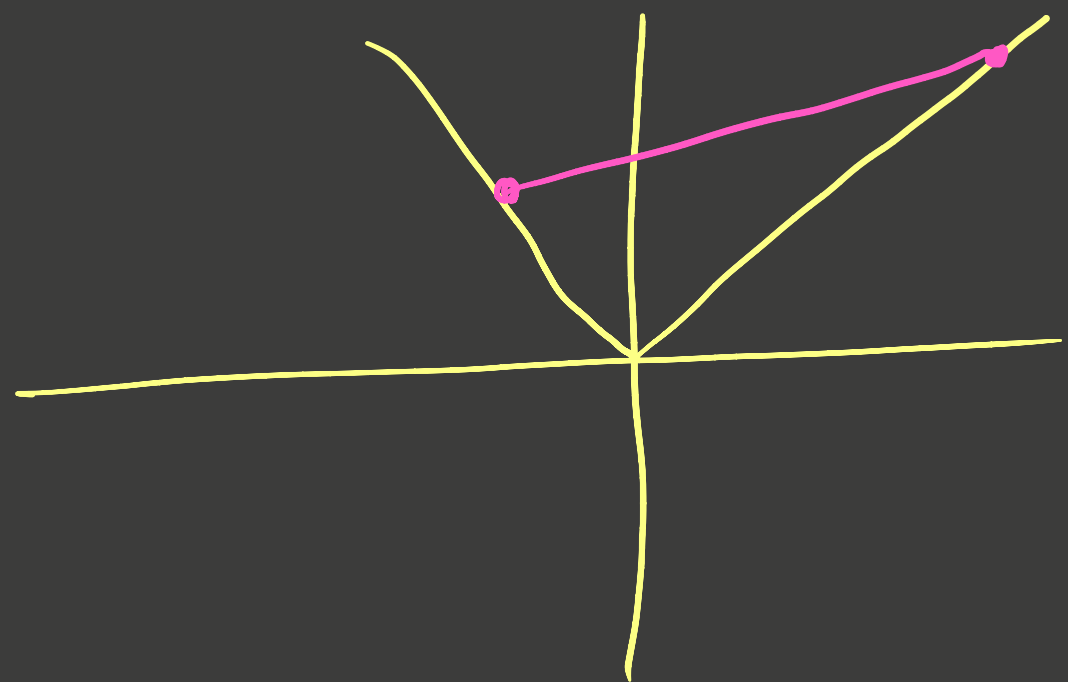


CONVEX

on $[-\infty, \infty)$

$\because$ LINE ABOVE CURVE

$y = |x|$

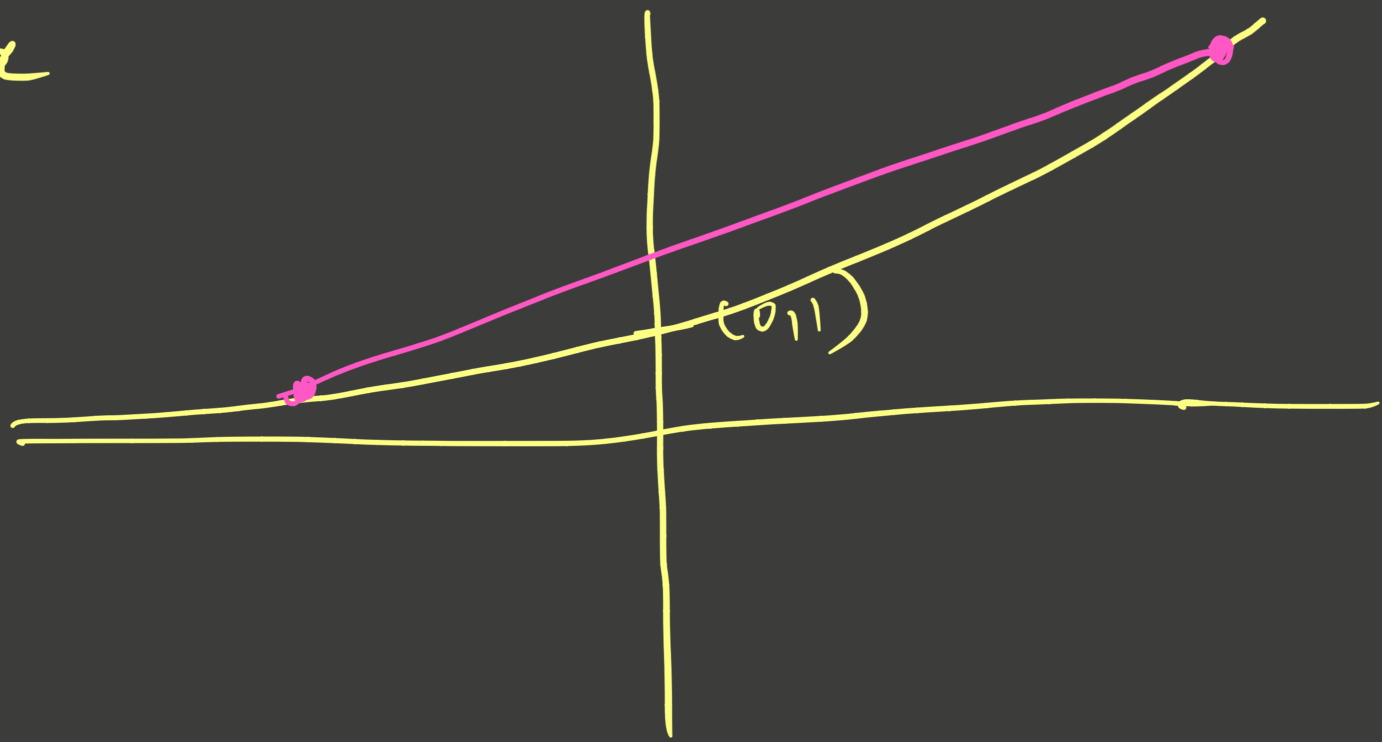


CONVEX

on $(-\infty, \infty)$

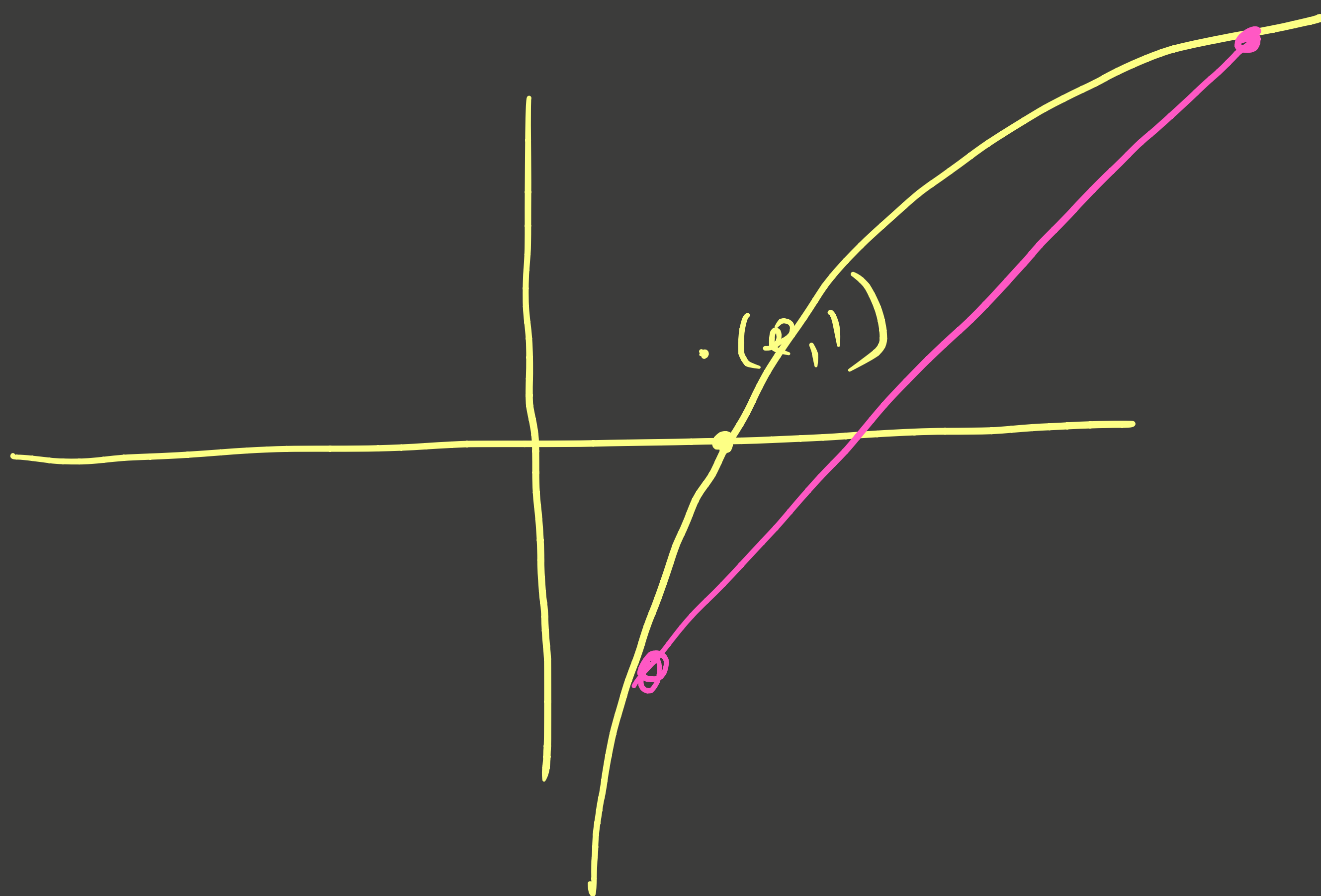
$\because$ LINE ABOVE CURVE

$y = e^x$



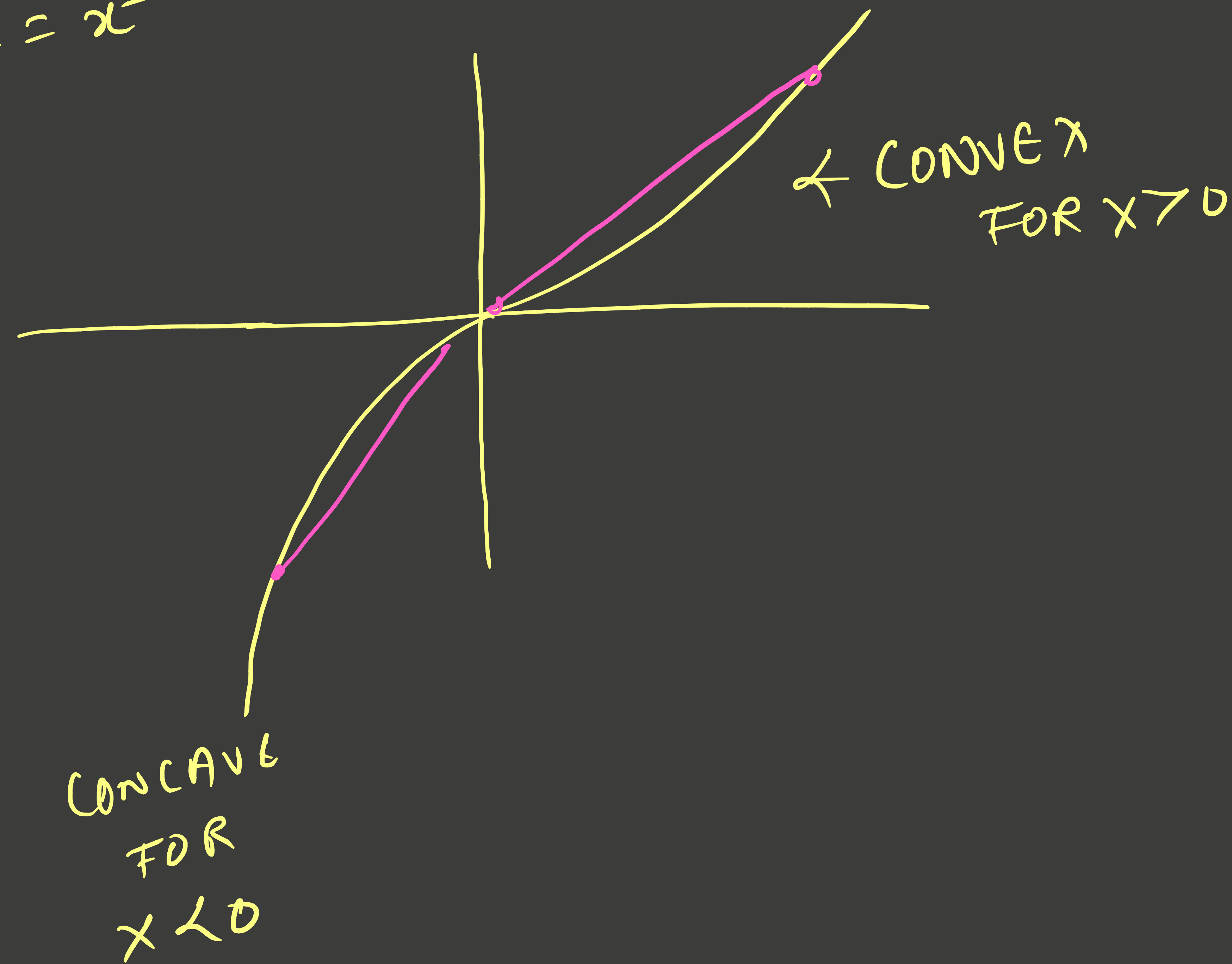
CONVEX

$y = \log_e x$



NON-CONVEX

$y = x^3$



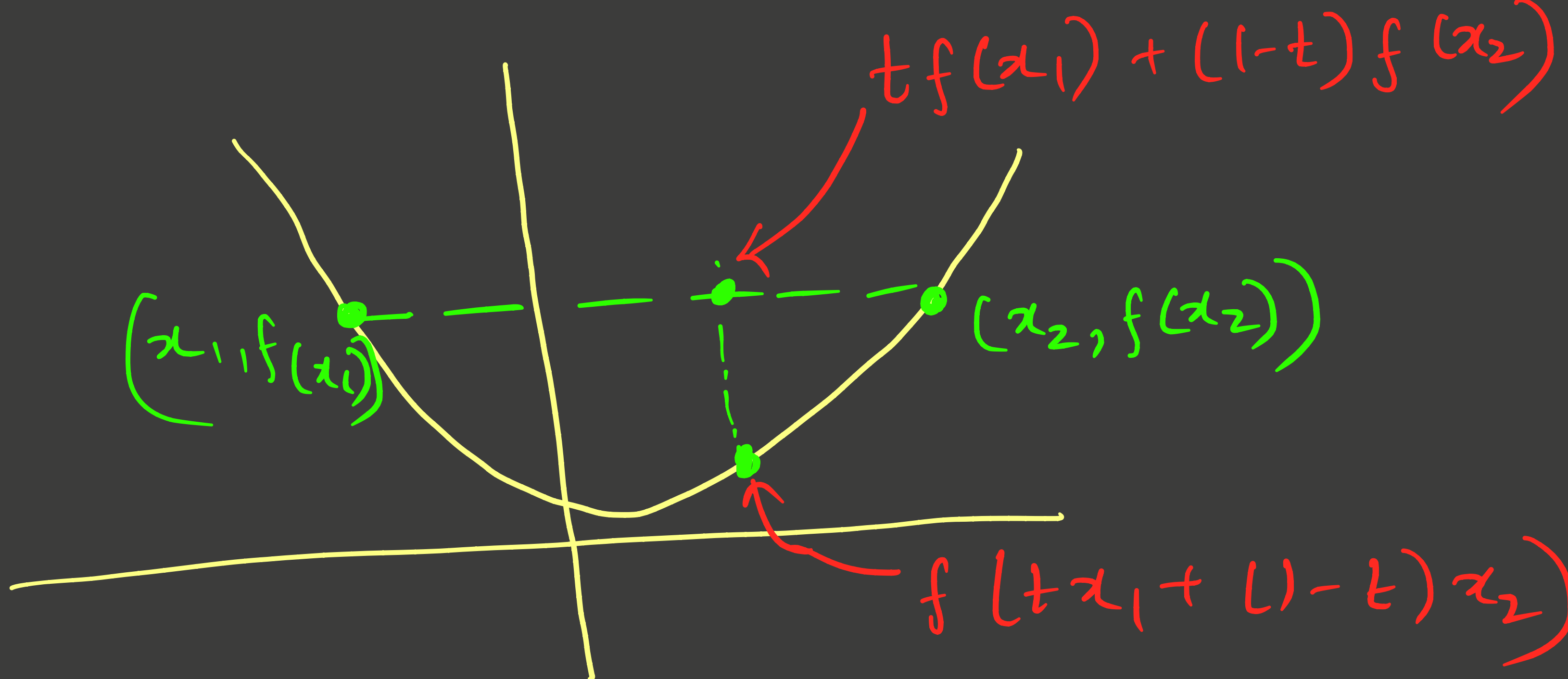
MATHEMATICALLY

f is CONVEX ON SET X

If

$\forall x_1, x_2 \in X$ AND $\forall t \in [0,1]$

$$\boxed{f(tx_1 + (1-t)x_2) \leq t f(x_1) + (1-t) f(x_2)}$$



Question. PROVE $f(x) = x^2$ is CONVEX

$$L.H.S. = f(tx_1 + (1-t)x_2)$$

$$= t^2x_1^2 + (1-t)^2x_2^2 + 2t(1-t)x_1x_2$$

$$R.H.S. = tf(x_1) + (1-t)f(x_2)$$

$$= tx_1^2 + (1-t)x_2^2$$

TO PROVE

LHS $\leq$ RHS

or $(t^2 - t)x_1^2 + [(1-t)^2 - (1-t)]x_2^2 + 2t(1-t)x_1x_2 \leq 0$

or $(t^2 - t)x_1^2 + (t^2 - t)x_2^2 - 2(t^2 - t)x_1x_2 \leq 0$

or $\underbrace{(t^2 - t)}_{\leq 0 \text{ for } t \in [0,1]}\underbrace{(x_1 - x_2)^2}_{\text{NON-NEG}} \leq 0$

HENCE PROVED

EASIER WAY

$\Rightarrow$ DOUBLE-DERIVATIVE TEST

$\hookrightarrow$ GENERALISE TO HESSIAN MATRICES

SOME PROPERTIES OF CONEX FUNCTIONS

① If $f(x)$ is CONVEX, then $K f(x)$ is CONVEX

② If $f(x), g(x)$ are CONVEX, then $f(x) + g(x)$ is CONVEX

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t) f(x_2)$$

$$g(tx_1 + (1-t)x_2) \le tg(x_1) + (1-t) g(x_2)$$

ADD

$$(f+g)(tx_1 + (1-t)x_2) \le t(f+g)(x_1) + (1-t)(f+g)(x_2)$$

HENCE PROVED

USING THIS

$(y - X\theta)^T (y - X\theta) + \theta^T\theta$ is CONVEX

" " " " " $+ |\theta|$ is CONVEX