KKT CONDITIONS

* USED FOR CONSTRAINED OPTMISATION OF THE FORM

MINIMIZE $f(x)$ where $x \in R^k$

s.t.

$h_i(x) = 0 \quad \forall\ i = 1, \ldots, m$ ← m equalities

$g_j(x) \leq 0 \quad \forall\ j = 1, \ldots, n$

STEP I

CREATE NEW FUNCTION

MULTIPLIERS FOR EQUALITY

'n' inequalities

$$L(x, \lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_n)$$

$$= f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{n} \mu_j g_j(x)$$

STEP II

minimize $L(x, \lambda, \mu)$ w.r.t. x or $\boxed{\nabla_x L(x, \lambda, \mu) = 0}$

This gives us 'k' equations ; $x \in R^k$

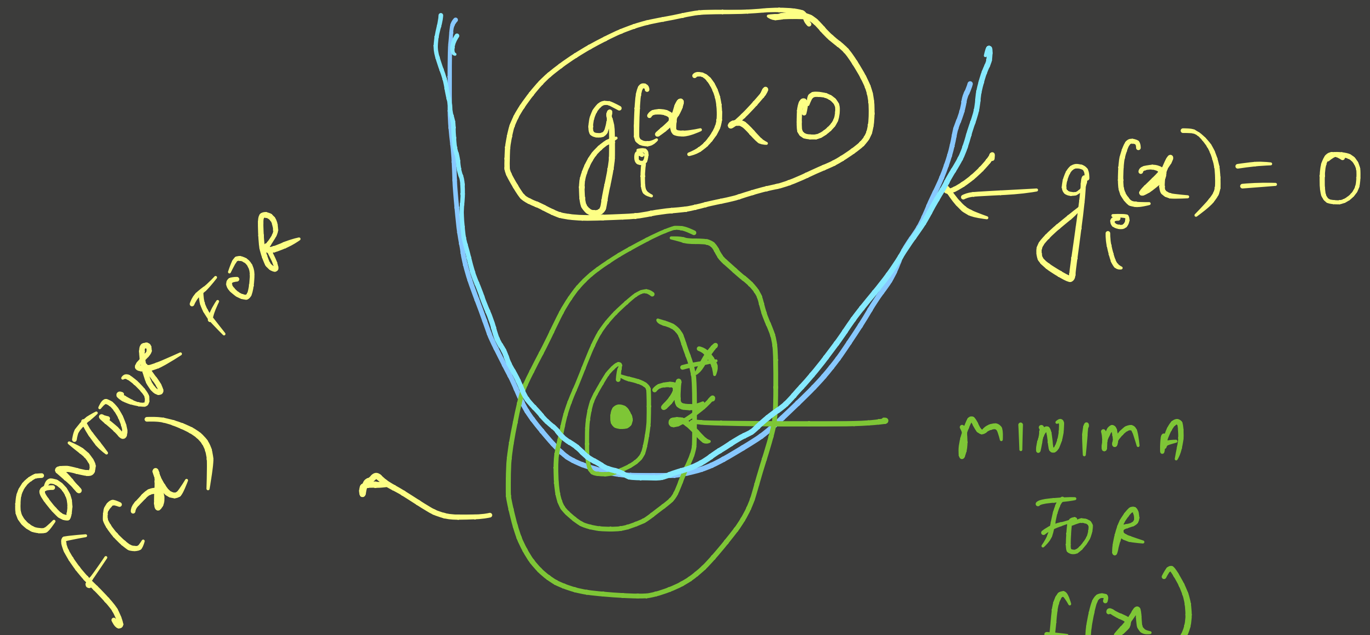
STEP III

$$\nabla_\lambda L(x, \lambda, \mu) = 0$$

Gives us 'm' equations

STEP IV

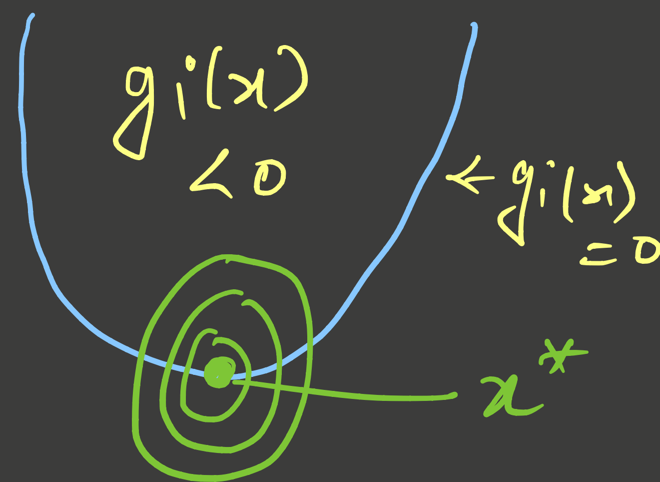
HOW TO HANDLE INEQUALITY CONSTRAINTS?



$\therefore\ g_i(x^*) < 0$

CONSTRAINT DOESN'T IMPACT x^* OR OPTIMA

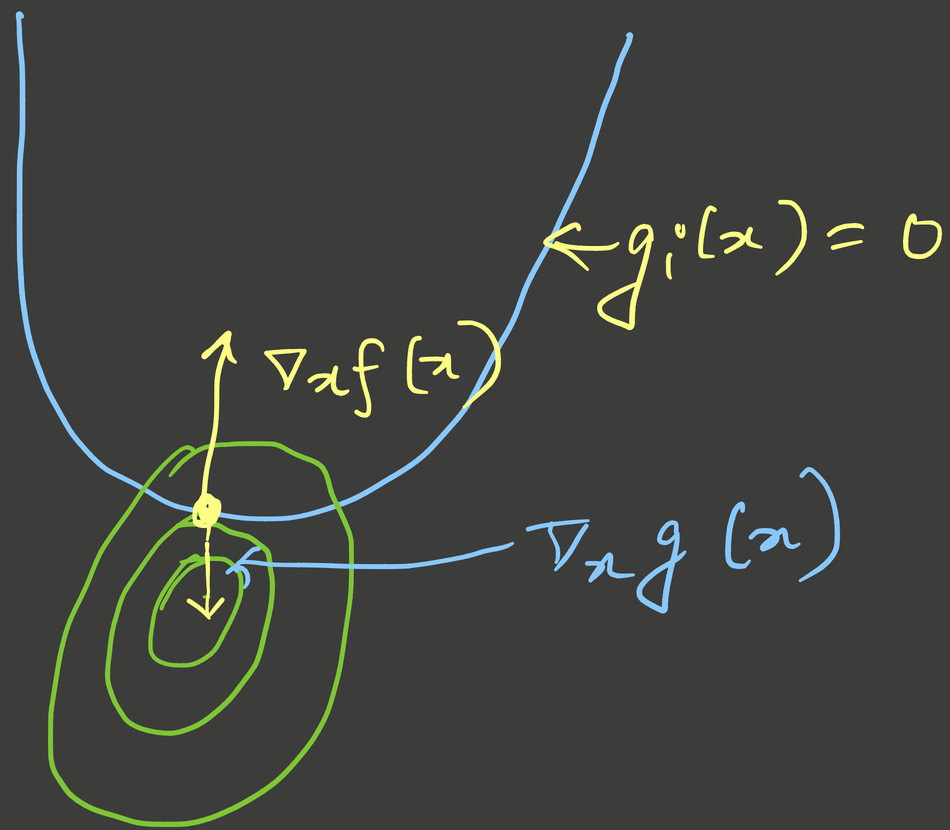
$\Rightarrow \mu_i = 0$



$g_i(x^*) = 0$

$$\boxed{\text{BOTH CASES: } \mu_i\, g_i(x^*) = 0}$$

CONSTRAINT ON μ_i's (MULTIPLIER FOR INEQUALITIES)



$$\min_x L(x, \lambda, \mu) \Rightarrow \nabla_x f(x) + \nabla_x \mu_i g_i(x) = 0$$

$$\Rightarrow 0 = \nabla f(x) + \mu_i \nabla g_i(x)$$

$$\Rightarrow \boxed{\mu_i = -\frac{\nabla f(x)}{\nabla g_i(x)} = +ve}$$

KKT CONDITIONS

* STATIONARITY (FOR MINIMIZATION)

$$\nabla_x f(x) + \sum_{i=1}^{m} \nabla_x \lambda_i h_i(x) + \sum_{i=1}^{n} \nabla_x \mu_i g_i(x) = 0$$

* EQUALITY CONSTRAINTS

$$\cancelto{0}{\nabla_x f(x)} + \sum_{i=1}^{m} \nabla_x \lambda_i h_i(x) + \sum_{i=1}^{n} \cancelto{0}{\nabla_x \mu_i g_i(x)} = 0$$

* INEQUALITY CONSTRAINTS (COMPLEMENTARY SLACKNESS)

$$\mu_i g_i(x) = 0 \quad \forall \; i = 1, \dots n$$

$$\mu_i \geq 0$$

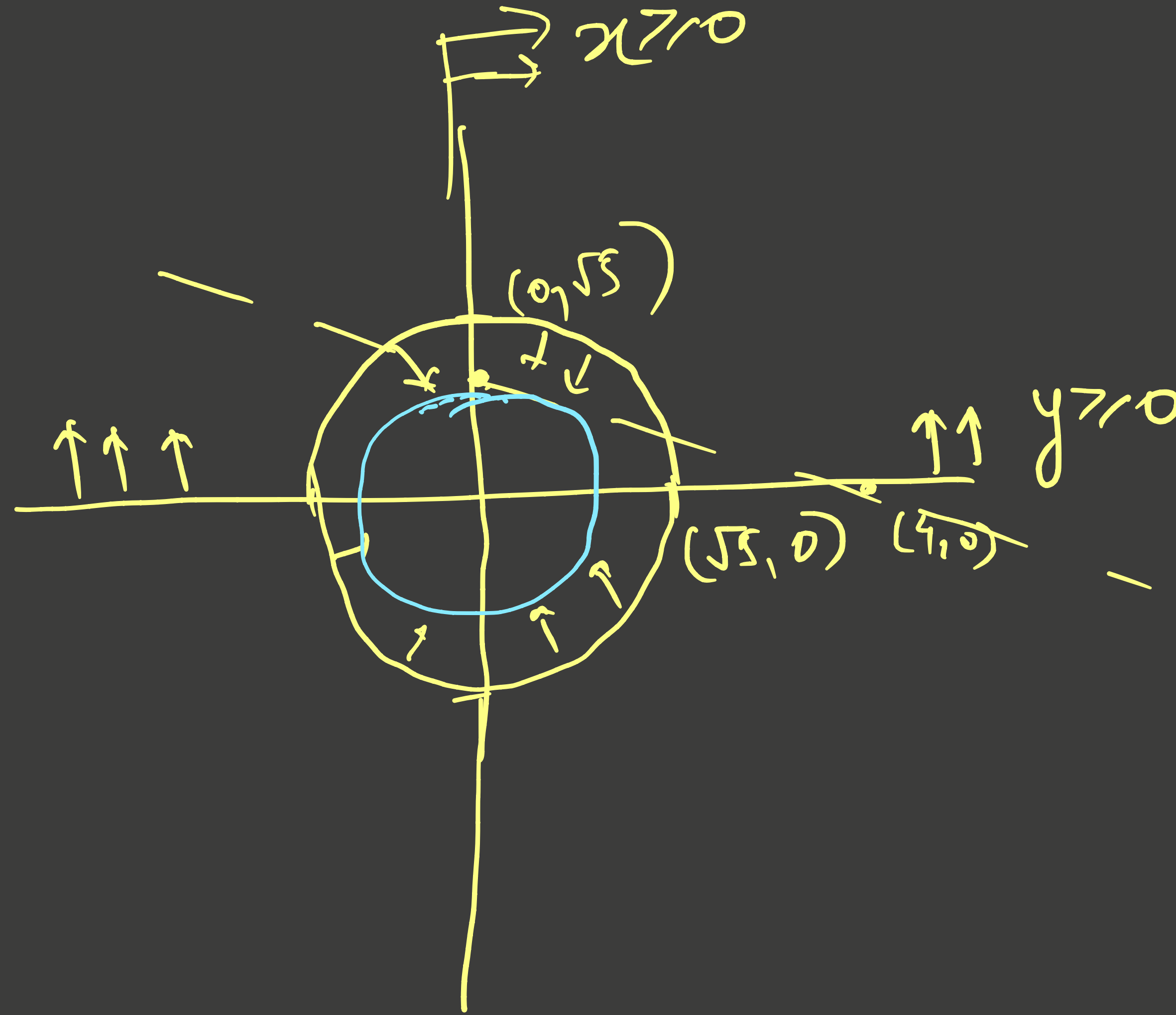
EXAMPLE 1

MINIMIZE x^2+y^2

S.t. $x^2+y^2 \leq 5$

$x+2y = 4$

$x, y \geq 0$



$f(x,y) = x^2+y^2$

$h(x,y) = x+2y-4$

$g_1(x,y) = x^2+y^2-5$

$g_2(x,y) = -x$

$g_3(x,y) = -y$

$L(x,y,\lambda,\mu_1,\mu_2,\mu_3)$

$= x^2+y^2+\lambda(x+y-3)$

$+ \mu_1(x^2+y^2-5)$

$+ \mu_2(-x) + \mu_3(-y)$

Stationarity

$$\nabla_x L(x, y, \lambda, \mu_1, \mu_2, \mu_3) = 0$$

$$\Rightarrow 2x + \lambda + 2\mu_1 x + (-\mu_2) = 0 \quad \ldots \text{(1)}$$

$$\nabla_y L(x, y, \lambda, \mu_1, \mu_2, \mu_3) = 0$$

$$\Rightarrow 2y + \lambda + 2\mu_1 y - \mu_3 = 0 \quad \ldots \text{(2)}$$

EQUALITY CONSTRAINTS

$$x + 2y = 4 \quad \ldots \text{(3)}$$

SLACKNESS

$$\mu_1 (x^2 + y^2 - 5) = 0 \quad \ldots \text{(4)}$$

$$\mu_1 = 0 \text{ or } x^2 + y^2 = 5$$

$$\mu_2 x = 0 \quad \cdots (5)$$

$\Rightarrow \mu_2 = 0$ or $x = 0$

If $x = 0 \Rightarrow y = 2 \Rightarrow x^2 + y^2 = 4 \ (<5)$

$$\mu_3 y = 0 \quad \cdots (6)$$

$\Rightarrow \mu_3 = 0$ or $y = 0$

$\Rightarrow x = 4$

$x^2 + y^2 \leq 5$ is violated

$\Rightarrow \boxed{y \neq 0 \text{ and } \mu_3 = 0}$

FROM (5) and (4),

$x = 0, y = 2$ gives SMALLER $x^2 + y^2$ than 5

$\therefore$ in (4) $\boxed{\mu_1 = 0}$