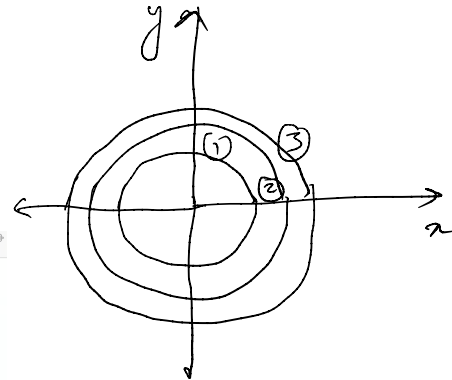


# Mathematics for Machine Learning - II

## CONTOUR PLOT

$$f(x, y) = x^2 + y^2$$

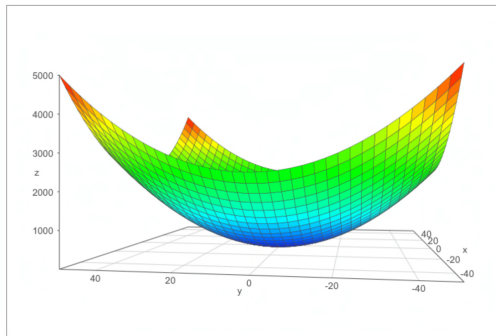
Plot  $f(x, y) = K$  for varying  $K$



4:23 PM Wed 23 Jan

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98%



Expression

$z = x^2 + y^2$

x Range (min, max)

-50, 50

y Range (min, max)

-50, 50

Resolution

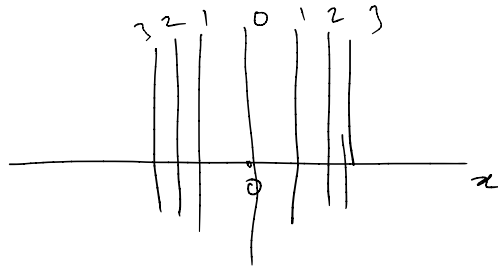
Calculate

This demo allows you to enter a mathematical expression in terms of  $x$  and  $y$ . When you hit the calculate button, the demo will calculate the value of the expression over the  $x$  and  $y$  ranges provided and then plot the result as a surface. The graph can be zoomed in by scrolling with your mouse, and rotated by dragging around. Clicking on the graph will reveal the  $x$ ,  $y$  and  $z$  values at that particular

You might also be interested in

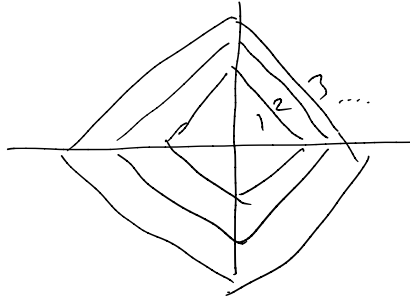
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Q) Draw contour plot for  $f(x) = x^2$



$$\text{Plot } x^2 = k$$

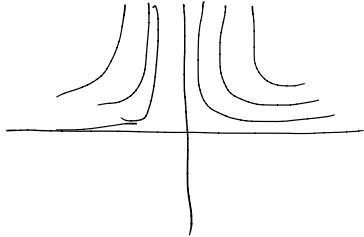
Q) Draw contour plot for for  $f(x, y) = |x| + |y|$



$$x, y \geq 0$$

$$\frac{x+y}{k} = 1$$

8) Draw contour for  $f(x,y) = x^2 y$



$$x^2 y = K$$

$$x^2 = \frac{K}{y}$$

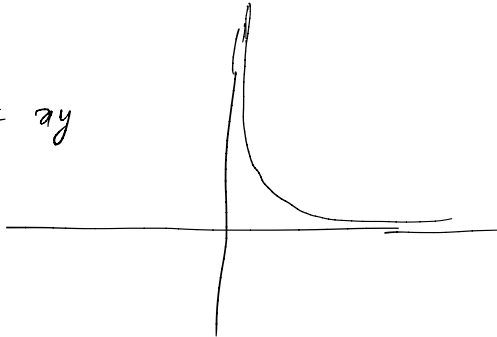
$$x y = \frac{K}{x}$$

$$K=1$$

$$y = \frac{1}{x^2}$$

9)

$$f(x,y) = xy$$



$$xy = K \Rightarrow x = \frac{K}{y}$$

$$K=1$$

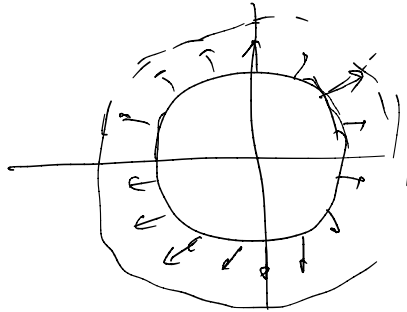
$$x = \frac{1}{y}$$

$$y = \frac{1}{x}$$

# Contour plots and (gradients)

↳ steepest change....

$$f(x,y) = x^2 + y^2$$



\* All points on contour have same  $f(x,y)$

\* which dir<sup>n</sup> do we move to increase  $f(x,y)$  now?

⇒ ⊥ to given point on the curve

gradient of:  $\nabla f(x,y)$

$$\begin{aligned} &= \begin{bmatrix} \frac{\partial}{\partial x} f(x,y) \\ \frac{\partial}{\partial y} f(x,y) \end{bmatrix} \\ &= \begin{bmatrix} 2x \\ 2y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

# Constrained Optimisation

Extrema  
(max or  
min)

$$f(x,y)$$

s.t

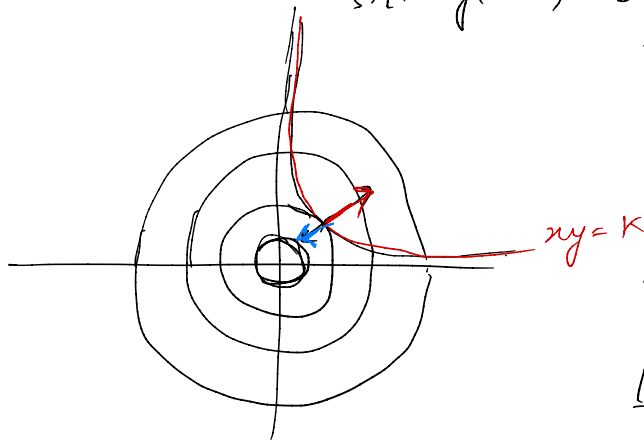
$$x^2 + y^2 = k$$

Q. Extrema is minima?  
or maxima?

more generally, ~~max~~ ~~min~~ extrema  $f(x, \dots)$   
s.t.  $g(x, \dots) = 0$

$\therefore$   $xy = 1$   
 $x$  vs  $y$  may be.

$\rightarrow$  gradient of  $f(x,y)$   
 $\rightarrow$  gradient of  $g(x,y)$



$$\nabla f(x,y) = \lambda (\nabla g(x,y))$$

$$\nabla f(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix} = \lambda \nabla g(x, y) = \lambda \begin{bmatrix} y \\ x \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2x = \lambda y \\ 2y = \lambda x \end{cases} \quad \left. \begin{array}{l} \text{3 variables} \quad \text{--- (1)} \\ 2 \text{ eq.} \quad \text{--- (2)} \end{array} \right\}$$

$$xy = K \quad \text{--- (3)}$$

① and ③

$$x = \frac{\lambda y}{2} \quad \& \quad xy = K \Rightarrow \frac{\lambda y}{2} \times y = K \Rightarrow \lambda y^2 = 2K$$

① and ②

$$4xy = \lambda xy \Rightarrow \lambda = 2$$

$$2x = 2y \Rightarrow x = y$$

$$\Rightarrow \begin{cases} x = y = \sqrt{K} \\ \lambda = 2 \end{cases}$$

# Constrained Optimisation

Extrema  
(max or  
minima)

$$f(x, y) = x^2 + y^2$$

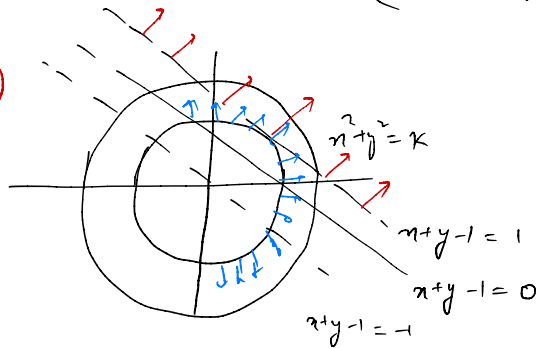
$$s.t. \quad x + y = 1$$

Q. Extrema is minima?  
or maxima?

more generally, we want to find extrema  $f(x, \dots)$   
s.t.  $g(x, \dots) = 0$

$\therefore x + y = 1$   
x and y may be.

→ Gradient of  $f(x, y)$   
→ Gradient of  $g(x, y)$



All points on contour  
have

$$\nabla f(x, y) = \lambda (\nabla g(x, y))$$

$$\nabla f(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix} = \lambda \nabla g(x, y) = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2x = \lambda \\ 2y = \lambda \end{cases} \quad \left. \vphantom{\begin{matrix} 2x = \lambda \\ 2y = \lambda \end{matrix}} \right\} \begin{array}{l} 2 \text{ eq's} \\ 1 \text{ variable} \end{array}$$

But we had

$$x + y - 1 = 0 \quad \dots \text{ 3rd eq.}$$

$$\therefore \begin{cases} 2x = \lambda \\ 2y = \lambda \end{cases} \Rightarrow x = y = \frac{\lambda}{2}$$

$$x + y = 1 \Rightarrow \lambda = 1 \Rightarrow x = y = \frac{1}{2}$$



x - Lagrangian multiplier

Lagrangian  $L(x, y, \lambda)$

$$\nabla L = 0$$

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial \lambda} = 0$$

Q) Find extrema for  $f(x, y) = x^2 y$   
 s.t.  $g(x, y) = x^2 + y^2 - 1 = 0$

$$L = x^2 y + \lambda (x^2 + y^2 - 1)$$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow 2xy + \lambda(2x) = 0$$

$$\frac{\partial L}{\partial y} \Rightarrow x^2 + \lambda(2y) = 0$$

$$\frac{\partial L}{\partial \lambda} \Rightarrow x^2 + y^2 - 1 = 0$$

$$\Rightarrow x(y + \lambda) = 0$$

$$x^2 + 2\lambda y = 0$$

$$x^2 + y^2 = 1$$

Case I

$$x = 0$$

$$\Rightarrow y^2 = 1 \Rightarrow y = \pm 1$$

$$\lambda = 0$$

$$f(x, y) = 0$$

Case II

$$x \neq 0$$

$$\Rightarrow y = -\lambda$$

$$\Rightarrow x^2 + 2\lambda(-\lambda) = 0$$

$$\Rightarrow 2\lambda^2 = 1$$

$$\Rightarrow x^2 = 2\lambda^2$$

$$x^2 + \lambda^2 = 1$$

$$\Rightarrow \lambda = \pm \frac{1}{\sqrt{2}}$$

$$x^2 = 2x^2 = \frac{2}{3}$$

$$y = \pm \sqrt{\frac{1}{3}}$$

$$\text{max } z^2 y = \frac{2}{3} \sqrt{\frac{1}{3}}$$

## KKT conditions

$$\text{Minimize } f(x)$$

$$\text{s.t. } h_i(x) = 0 \quad \forall i = 1, \dots, m$$

$$g_j(x) \leq 0 \quad \forall j = 1, \dots, n$$

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^m \lambda_i h_i(x) + \sum_{j=1}^n \mu_j g_j(x)$$

Now, if  $g_i(x^*) < 0$ , then  $\mu_i$  can be set to zero (constraint doesn't part opt)

$$\text{else } g_i(x^*) = 0$$

$$\therefore \boxed{\mu_i g_i(x^*) = 0}$$

$$\mu_i > 0$$

Stationarity

$$\nabla_{\alpha} f(x) + \sum_{i=1}^m \nabla_{\alpha} \lambda_i h_i(x) + \sum_{i=1}^n \nabla_{\alpha} \mu_i g_i(x) = 0$$

Equality

$$\nabla_{\lambda} f(x) + \sum_{i=1}^m \nabla_{\lambda} \lambda_i h_i(x) + \sum_{i=1}^n \nabla_{\lambda} \mu_i g_i(x) = 0$$

Inequality / complementary slackness

$$\begin{aligned} \mu_i g_i(x) &= 0 \\ \mu_i &\geq 0 \end{aligned}$$