Differential Entropy

* Extension of entropy for continuous r.v.s.

* Let X be r.v. with pdf f whose support is $\mathcal{X}$.

  Differential entropy $h(X) = -\int_{\mathcal{X}} f(x) \log f(x)\, dx$

Differential Entropy

Differential entropy $h(X) = -\int_{\mathcal{X}} f(x) \log f(x)\, dx$

UNIFORM DISTRIBUTION

$f(x) = \frac{1}{b-a}$; $\mathcal{X} = [a, b]$

$$h(x) = -\int_a^b \frac{1}{b-a} \log\left(\frac{1}{b-a}\right) dx = -\left[\left(\frac{1}{b-a}\right) \log \frac{1}{b-a}\, x\right]_a^b$$

$$= -\left[\left(\frac{1}{b-a}\right)(b-a) \log \frac{1}{b-a}\right]$$

$$= \log(b-a)$$

Differential Entropy

Differential entropy $h(X) = -\int_{\mathcal{X}} f(x) \log f(x)\, dx$

1D NORMAL DISTRIBUTION

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad : \mathcal{X} = [-\infty, \infty]$$

$$h(X) = -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

Differential Entropy

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$$h(X) = -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \left[\log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \left(\frac{(x-\mu)^2}{2\sigma^2}\right)\right] dx$$

$$= \log_e\left(\sigma\sqrt{2\pi e}\right) = \frac{1}{2}\log\left(2\pi e\,\sigma^2\right)$$

Differential Entropy

Differential entropy $h(X) = -\int_{\mathcal{X}} f(x) \log f(x)\, dx$

MULTIVARIATE GAUSSIAN

$$X \sim N_n(\mu, \Sigma)$$

$$h(X) = \frac{1}{2} \log \left[(2\pi e)^n |\Sigma| \right]$$

Differential Entropy

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MULTIVARIATE GAUSSIAN

$$X \sim N_n(\mu, \Sigma)$$

$$h(X) = \frac{1}{2} \log \left[(2\pi e)^n |\Sigma| \right]$$

e.g. $X \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$

$$h_1(X) = \frac{1}{2} \log \left[(2\pi e)^2 (1) \right]$$

Differential Entropy

Differential entropy $h(X) = -\int_{\mathcal{X}} f(x) \log f(x)\, dx$

MULTIVARIATE GAUSSIAN

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$$h(X) = \frac{1}{2} \log \left[(2\pi e)^2 (1) \right]$$

$X_2 \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \right)$

$$h(X_2) = \frac{1}{2} \log \left[(2\pi e)^2 (0.75) \right]$$

Differential Entropy

Differential entropy $h(X) = -\int_X f(x) \log f(x)\, dx$

$X_1 \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

$h(X_1) = \frac{1}{2} \log \left[(2\pi e)^2 (1)\right]$

$X_2 \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}\right)$

$h(X_2) = \frac{1}{2} \log \left[(2\pi e)^2 0.75\right]$

$2h(X_2) - 2h(X_1) = \log \left[\frac{(2\pi e)^2}{(2\pi e)^2} \frac{(0.75)}{1}\right]$

$= \log (0.75) = -0.28$

$\Rightarrow$ More entropy in X_1

CONDITIONAL DIFFERENTIAL ENTROPY

let X and Y be continuous rvs with joint pdf $f(x,y)$

Conditional Entropy

$$h(X|Y) = -\int_{x,y} f(x,y) \log f(x|y)\, dx\, dy$$

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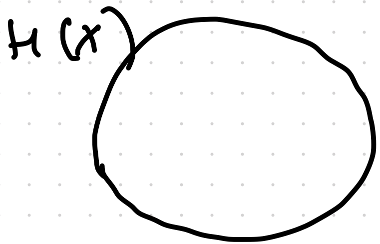
Conditional Entropy

$$h(X|Y) = -\int_{x,y} f(x,y) \log f(x|y)\, dx\, dy$$

$$h(X|Y) = h(X,Y) - h(Y)$$

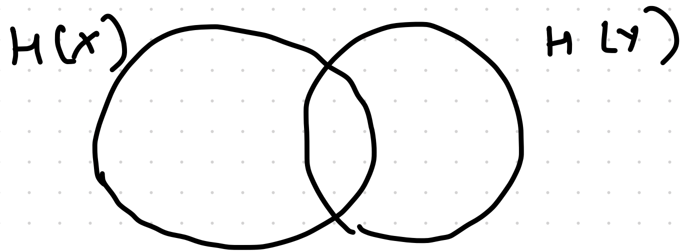
CONDITIONAL DIFFERENTIAL ENTROPY



$H \rightarrow$ Discrete

$h \rightarrow$ Continuous

CONDITIONAL DIFFERENTIAL ENTROPY



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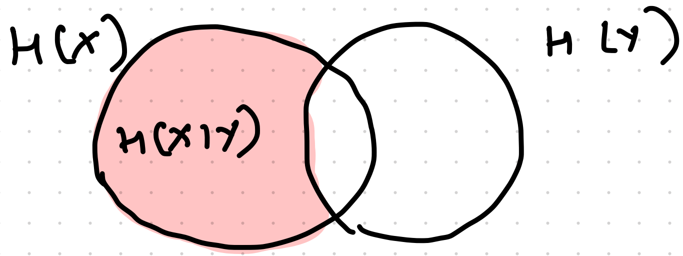

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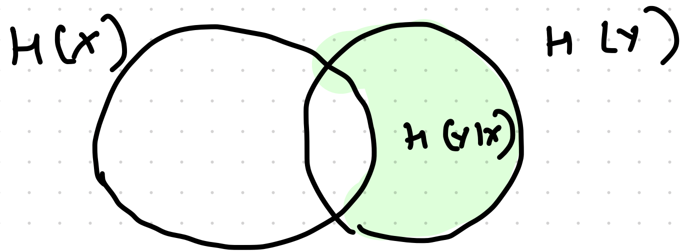

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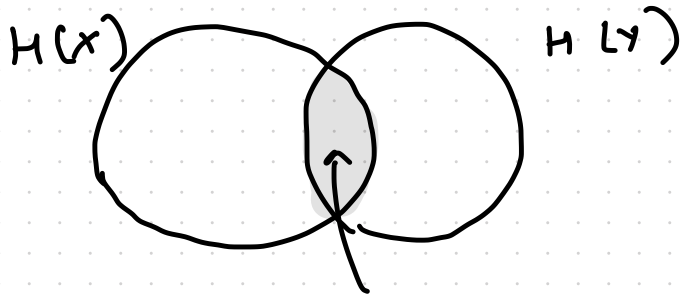


$I(X;Y)$

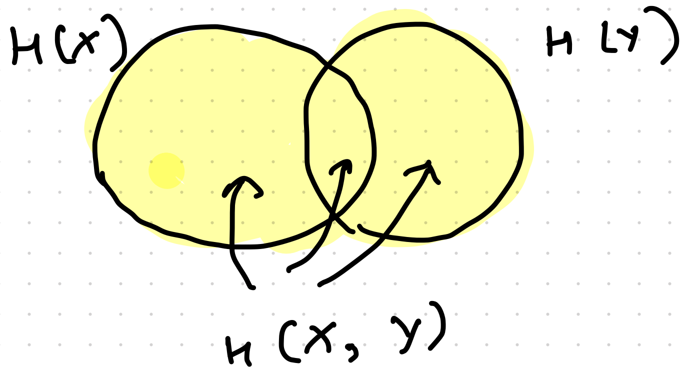
MUTUAL INFORMATION

$= H(X) - H(X|Y)$

$\boxed{I(X;Y) = H(Y) - H(Y|X)}$ Discrete

$\boxed{I(X;Y) = h(y) - h(y|x)}$ Continuous

CONDITIONAL DIFFERENTIAL ENTROPY



CONDITIONAL DIFFERENTIAL ENTROPY

Let $X \sim N_2\left(0, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$

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$$h(X) = h(X_1, X_2) = \frac{1}{2} \log \left[(2\pi e)^n |\Sigma|\right]$$

$$= \frac{1}{2} \log \left((2\pi e)^2 (1-\rho^2)\right)$$

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$$h(X_1) = \frac{1}{2} \log(2\pi e) = h(X_2)$$

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$$h(X_1 | X_2) = h(X_1, X_2) - h(X_2) = h(X) - h(X_2)$$

$$= \frac{1}{2} \log \frac{(2\pi e)^2 (1-\rho^2)}{(2\pi e)}$$

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$$I(X_1; X_2) = h(X_1) - h(X_1 | X_2)$$

$$= \frac{1}{2} \log 2\pi e - \frac{1}{2} \log 2\pi e (1-\rho^2) = \frac{-1}{2} \log(1-\rho^2)$$

CONDITIONAL DIFFERENTIAL ENTROPY

Let $X \sim N_2\left(0, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$

$$I(X_1;X_2) = h(X_1) - h(X_1 \mid X_2)$$
$$= \frac{1}{2}\log 2\pi e - \frac{1}{2}\log 2\pi e(1-\rho^2) = \frac{-1}{2}\log(1-\rho^2)$$

$\rho \to \pm 1 \Rightarrow I(X_1;X_2) \to \infty$

Perfect correlatⁿ: M.I. $\to \infty$

CONDITIONAL DIFFERENTIAL ENTROPY

Let $X \sim N_2\left(0, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$

$$I(X_1;X_2) = h(X_1) - h(X_1 | X_2)$$
$$= \frac{1}{2}\log 2\pi e - \frac{1}{2}\log 2\pi e(1-\rho^2) = -\frac{1}{2}\log(1-\rho^2)$$

$\rho \to \pm 1 \Rightarrow I(X_1;X_2) \to \infty$

Perfect Correlat^n: M.I. $\to \infty$

$\rho \to 0 \Rightarrow I(X_1;X_2) \to 0$ No correlat^n : M.I. $\to 0$