Lagrangian and Duality

Nipun Batra

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IIT Gandhinagar

Lectures heavily inspired by the Maths for Machine learning book

• Minimax inequality states: $\max_{\mathbf{y}} \min_{\mathbf{x}} q(\mathbf{x}, \mathbf{y}) \leqslant \min_{\mathbf{x}} \max_{\mathbf{y}} q(\mathbf{x}, \mathbf{y})$

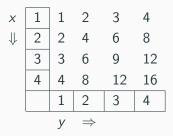
- Minimax inequality
 states:max_y min_x q(x, y) ≤ min_x max_y q(x, y)
- We first prove For all $x, y = \min_{x} q(x, y) \leqslant \max_{y} q(x, y)$

• Let us choose q(x, y) = xy

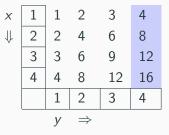
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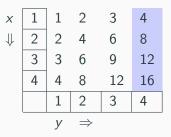


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- y = 4 maximizes $q(x, y) \forall x$



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X	1	1	2	3	4
\Downarrow	2	2	4	6	8
	3	3	6	9	12
	4	4	8	12	16
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- x = 1 minimizes $q(x, y) \forall y$

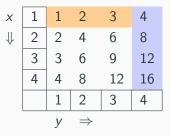
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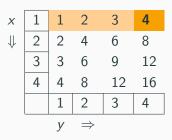


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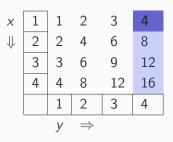
- We just showed For all $x, y = \min_{x} q(x, y) \leqslant \max_{y} q(x, y)$
- The equality occurs at x = 1, y = 4



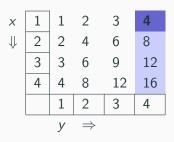
• Let us now find $\max_{y} \min_{x} q(x, y)$



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- We can thus see our Minimax inequality $\max_{\mathbf{y}} \min_{\mathbf{x}} q(\mathbf{x}, \mathbf{y}) \leqslant \min_{\mathbf{x}} \max_{\mathbf{y}} q(\mathbf{x}, \mathbf{y})$



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This would give infinte penalty if constraint is not satisfied. But, this formulation is hard to solve too.

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When $\lambda \geqslant 0$, the Lagrangian $\mathcal{L}(x,\lambda)$ is a lower bound of J(x). Hence, the maximum of $\mathfrak{L}(x,\lambda)$ with respect to λ is

$$J(\mathbf{x}) = \max_{\mathbf{\lambda} \geqslant 0} \mathfrak{L}(\mathbf{x}, \mathbf{\lambda})$$

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We can write the dual objective as a function of λ as

$$\mathfrak{D}(\lambda) = \min_{\mathbf{x} \in \mathbb{R}^d} \mathfrak{L}(\mathbf{x}, \lambda)$$

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- For SVM like formulations, primal objective is the same as dual objective (strong duality)
- For some problems, there is a "daulity-gap" between the two objectives