SVM Soft Margin Classification

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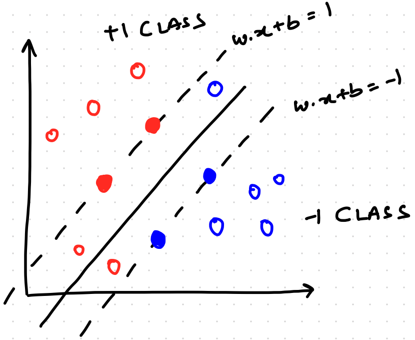
"SLIGHTLY" NON-SEPARABLE DATA



Soft-Margin SVM

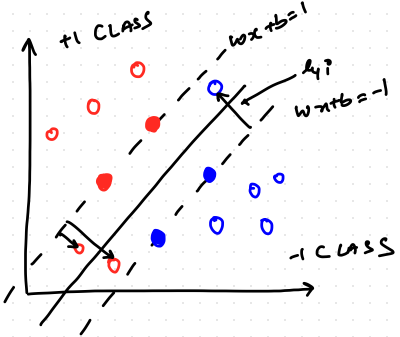
- Can we learn SVM for “slightly” non-separable data without projecting to a higher space?

Soft-Margin SVM

- Can we learn SVM for “slightly” non-separable data without projecting to a higher space?
- Introduce some “slack” (ξ_i) or loss or penalty for samples - allow some samples to be misclassified

“SLIGHTLY” NON-SEPARABLE DATA

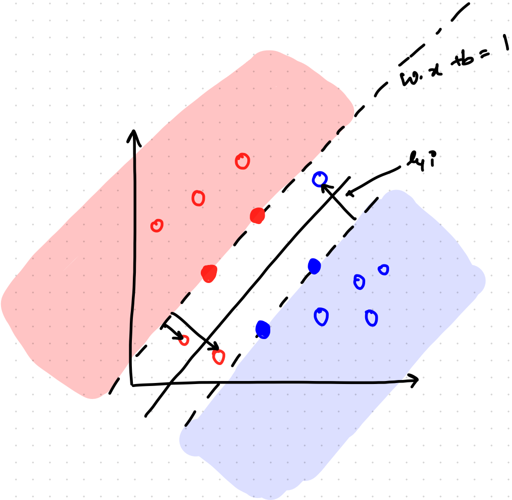
ξ_i: Distance from margin



ZONE 1

$y_i = +1$

$w \cdot x_i + b \geq 1$



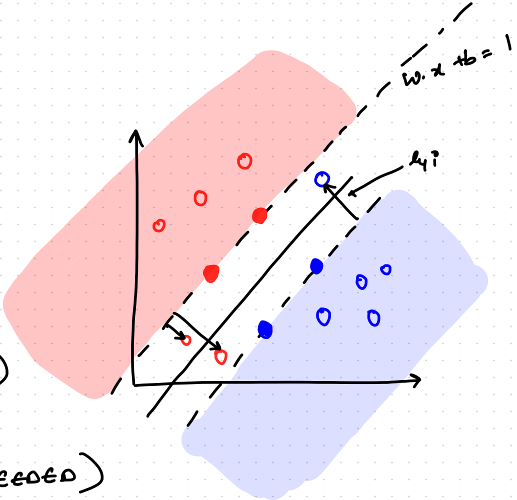
ZONE 1

$y_i(\vec{w}\cdot\vec{x_i}+b) > 1$

$Loss_i = 0$ ($\xi_i = 0$)

⇓

NO PENALTY
(OR SLACK NEEDED)

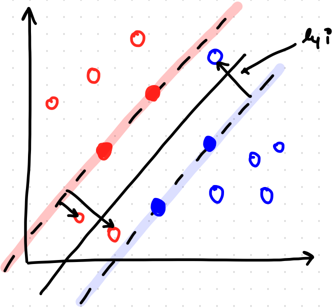


ZONE 2

$y_i (\vec{w} \cdot \vec{x}_i + b) = 1$

$LOSS_i = 0$

$(\xi_i = 0)$



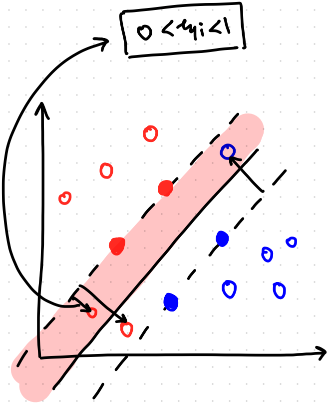
ZONE 3

$y_i(\vec{w}\cdot\vec{x}_i + b) < 1$

$LOSS_i \neq 0$ $(0 < \xi_i < 1)$

POINT CORRECTLY CLASSIFIED

(BUT WRONG SIDE OF MARGIN)



ZONE 3

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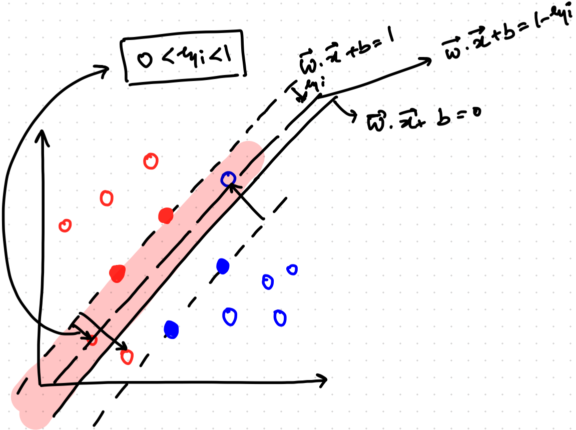


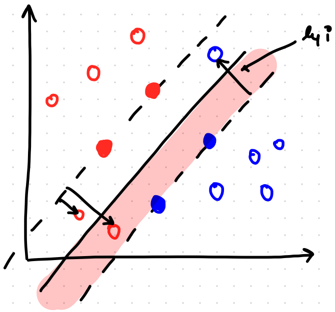
ZONE 4

$y_i(\vec{w} \cdot \vec{x_i} + b) < 1$

POINT INCORRECTLY CLASSIFIED

$LOSS_i \neq 0$

$\xi_i > 1$



ZONE 4

$y_i(\vec{w}\cdot\vec{x_i}+b) < 1$

POINT INCORRECTLY CLASSIFIED

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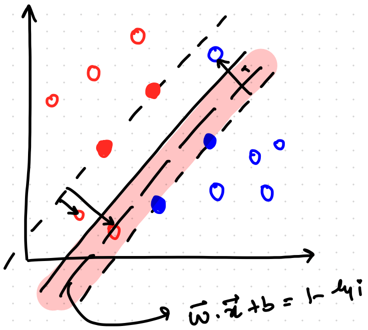
$\xi_i > 1$



Soft-Margin SVM

Change Objective

$\min \frac{1}{2}||\bar{w}||^2 + C\sum_{i=1}^{n}\xi_i$

s.t. $y_i(\bar{w}\bar{x}_i + b) \geq 1 - \xi_i$

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s.t. $y_i(\bar{w}\bar{x}_i + b) \geq 1 - \xi_i$

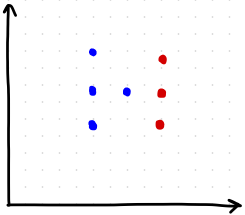
In Dual:

$$\text{Minimize} \sum_{i=1}^{n}\alpha_i - \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j y_i y_j \bar{x}_i \bar{x}_j$$

s.t.

$$0 \leq \alpha_i \leq C \quad \& \quad \sum_{i=1}^{n}\alpha_i y_i = 0$$

BIAS-VARIANCE TRADE-OFF



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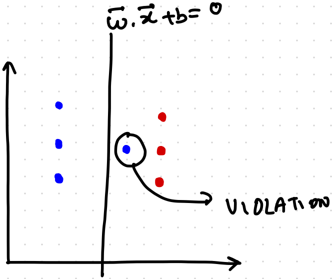
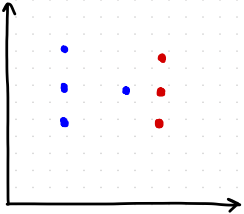
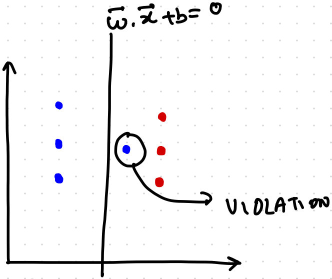




LOW C
LOW PENALTY FOR VIOLATION
HIGH TRAIN ERROR
HIGH BIAS

BIAS-VARIANCE TRADE-OFF



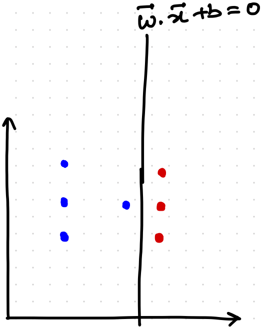
LOW C

LOW PENALTY FOR VIOLATION

HIGH TRAIN ERROR

HIGH BIAS

BIG MARGIN



HIGH C

HIGH PENALTY FOR VIOLATION

HIGH VARIANCE

SMALL MARGIN

Bias Variance Trade-off for Soft-Margin SVM

Low C $\implies$ Higher train error (higher bias)

High C $\implies$ Very sensitive to datasete (high variance)

Soft-Margin SVM

If $C \to 0$

Objective $\to \min \frac{1}{2}||\bar{w}||^2$

$\implies$ Choose large margin (without worrying for ξ_is)

Recall: Margin = $\frac{2}{||\bar{w}||}$

If $C \to \infty$ (or very large) Objective $\to \min C \sum \xi_i$ or choose W, b, s.t. ξ_i is small!

Soft-Margin SVM

Q) What is the equivalent of hard margin?

a $C \to 0$

b $C \to \infty$

Soft-Margin SVM

Q) What is the equivalent of hard margin?

a $C \to 0$

b $\boxed{C \to \infty} \implies$ No violations!!

Soft-Margin SVM

Types of support vectors:

- Zone 2: $y_i(\bar{w}\bar{x}_i + b) = 1$
- Zone 3: $0 < \xi_i < 1$ (correctly classified)
- Zone 4: $\xi_i > 1$ (Misclassified)

$\therefore$ As C increases, # support vectors decreases

Notebook: SVM-soft-margin

SVM Formulation in the Loss + Penalty Form

Objective:

$$\min \frac{1}{2}||\bar{w}||^2 + C\sum_{i=1}^{N}\xi_i$$

Now:

$$y_i(\bar{w}\bar{x}_i + b) \geq 1 - \xi_i$$

$$\xi_i \geq 1 - y_i(\bar{w}\bar{x}_i + b)$$

But $\xi_i \geq 0$

$$\therefore \xi_i = \max\left[0, 1 - y_i(\bar{w}\bar{x}_i + b)\right]$$

SVM Formulation in the Loss + Penalty Form

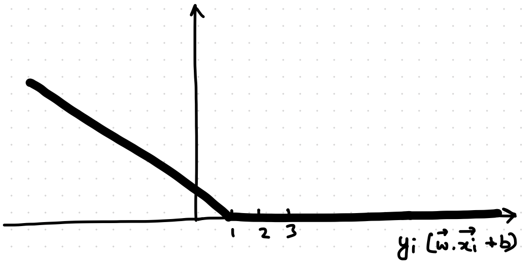
$\therefore$ Objective is:

$$\min C \sum \xi_i + \frac{1}{2}||\bar{w}||^2$$

$$\implies \min C \sum_{i=1}^{N} \max\left[0, 1 - y_i(\bar{w}\bar{x}_i + b)\right] + \frac{1}{2}||\bar{w}||^2$$

$$\implies \min \underbrace{\sum_{i=1}^{N} \max\left[0, 1 - y_i(\bar{w}\bar{x}_i + b)\right]}_{\text{Loss}} + \underbrace{\frac{1}{2C}||\bar{w}||^2}_{\text{Regularisation}}$$

HINGE LOSS



Loss Function for Sum (Hinge Loss)

Loss function is $\sum_{i=1}^{N} \max\left[0, 1 - y_i(\bar{w}\bar{x}_i + b)\right]$

- Case I $\qquad y_i(\bar{w}\bar{x}_i + b) = 1$
  Lies on Margin: $Loss_i$ = 0

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Loss function is $\sum_{i=1}^{N} \max\left[0, 1 - y_i(\bar{w}\bar{x}_i + b)\right]$

- Case I $\quad y_i(\bar{w}\bar{x}_i + b) = 1$
  Lies on Margin: $Loss_i$ = 0
- Case II
  $$y_i(\bar{w}\bar{x}_i + b) > 1$$
  $Loss_i = 0$

Loss Function for Sum (Hinge Loss)

Loss function is $\sum_{i=1}^{N} \max\left[0, 1 - y_i(\bar{w}\bar{x}_i + b)\right]$

- Case I $\quad y_i(\bar{w}\bar{x}_i + b) = 1$
  Lies on Margin: $Loss_i$ = 0
- Case II
  $$y_i(\bar{w}\bar{x}_i + b) > 1$$
  $Loss_i = 0$
- Case III
  $$y_i(\bar{w}\bar{x}_i + b) < 1$$
  $Loss_i \neq 0$

Hinge Loss Continued

Q) Is hinge loss convex and differentiable?

Convex: ✓

Differentiable: X

Subgradient: ✓

SVM Loss is Convex

Hinge Loss $\sum(\max[0, (1 - y_i(\bar{w}x_i + b))]$ is convex

Penalty $\frac{1}{2}||\bar{w}||^2$ is convex

$\therefore$ SVM loss is convex