Support Vector Machines

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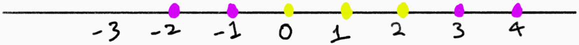
Non-Linearly Separable Data



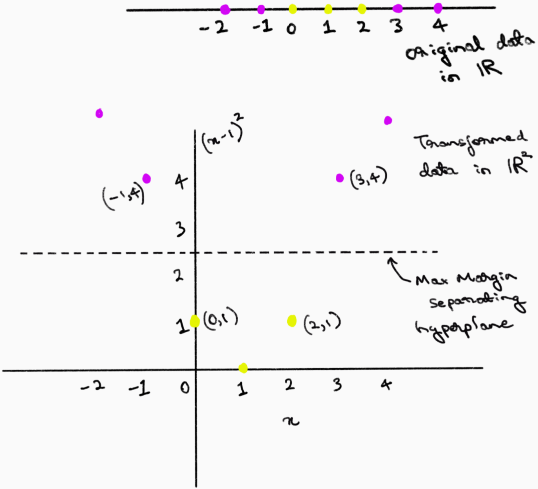
Data not separate in $\mathbb{R}$

Can we still use SVM?

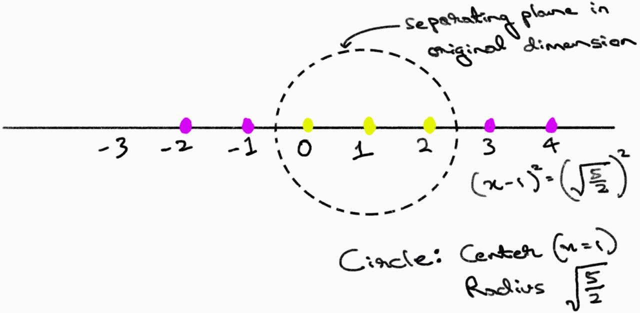
Yes!

How? Project data to a higher dimensional space.

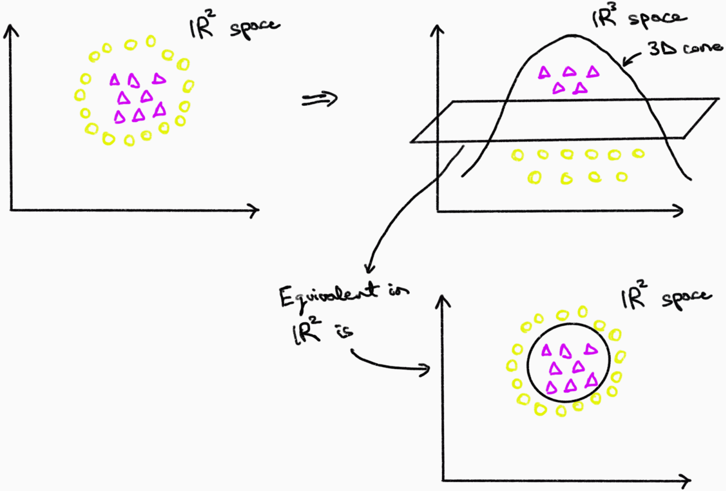
Non-Lineary Separable Data



Non-Lineary Separable Data



Another Example Transformation



Projection/Transformation Function

$\phi : \mathbb{R}^d \to \mathbb{R}^D$

where, d = original dimension

D = new dimension

In our example:

$d = 1; D = 2$

Linear SVM:

Maximize

$$L(\alpha) = \sum_{i=1}^{N} \alpha_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j y_i y_j \overline{x_i} . \overline{x_j}$$

such that constriants are satisfied.

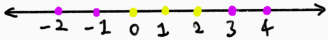
$$\downarrow$$

Transformation (ϕ)

$$\downarrow$$

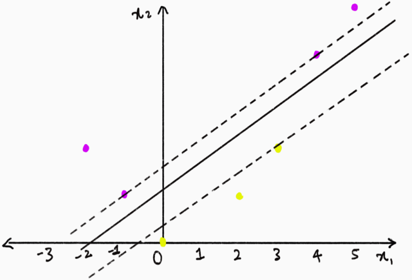
$$L(\alpha) = \sum_{i=1}^{N} \alpha_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j y_i y_j \phi(\overline{x_i}) . \phi(\overline{x_j})$$

Trivial Example (Again)



Original data $(x) \in R$

Transformed Data $\left(\phi(x) = \langle \sqrt{2}x, x^2 \rangle\right)$



Steps

1. Compute $\phi(x)$ for each point

$$\phi : \mathbb{R}^d \to \mathbb{R}^D$$

2. Compute dot products over $\mathbb{R}^D$ space

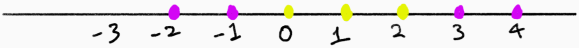
Q. If $D >> d$

Both steps are expensive!

Kernel Trick

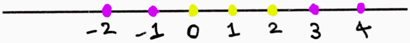
Can we compute $K(\bar{x}_i, \bar{x}_j)$
s.t.
$K(\bar{x}_i, \bar{x}_j) = \phi(\bar{x}_i).\phi(\bar{x}_j)$
where,
$K(\bar{x}_i, \bar{x}_j)$ is some function of dot product in original dimension
$\phi(\bar{x}_i).\phi(\bar{x}_j)$ is dot product in high dimensions (after transformation)

Kernel Trick



$\phi(x) =< \sqrt{2}x, x^2 >$
$K(x_i, x_j) = (1 + x_i x_j)^2 - 1$ where $x_i x_j$ is dot product in lower dimensions

$$\begin{aligned}(1 + x_i x_j)^2 - 1 &= 1 + 2x_i x_j + x_i^2 x_j^2 - 1 \\ &=< \sqrt{2}x_i, x_i^2 > . < \sqrt{2}x_j, x_j^2 > \\ &= \phi(x_i).\phi(x_j)\end{aligned}$$



Original Dataset

#	x	y
1	-2	-1
2	-1	-1
3	0	1
⋮	⋮	⋮

Transformed Dataset

#	$\sqrt{2}x$	x^2	y
1	$-2\sqrt{2}$	4	1
2	$-\sqrt{2}$	1	-1
3	0	0	1
⋮	⋮	⋮	⋮

$\phi(x_1) =< -2\sqrt{2}, 4 >; \phi(x_2) =< -\sqrt{2}, 1 >$ Transformation

$\phi(x_1)\phi(x_2) = -2\sqrt{2} \times -\sqrt{2} + 4 \times 1 = 8$ Dot product in 2D

$K(x_1, x_2) = \{1 + (-2) \times (-1)\}^2 - 1$ Dot product in 1D

Kernel Trick

Q) Why did we use dual form?
Kernels again!!

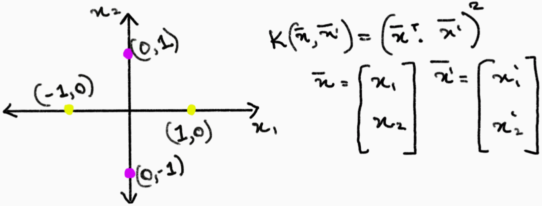
Primal form doesn't allow for the kernel trick
$K(\bar{x}_1, \bar{x}_2)$ in dual and compute $\phi(x)$ and then dot product in D dimensions

Gram Matrix: (Positive Semi-Definite)

$K(x_i, x_j) = (1 + x_i x_j)^2$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	24	8	0	0	8	24	48
x_2	8	1	0	-1	0	...	
x_3	0	...	...	...	...	...	...
x_4	0						
x_5	8						
x_6	24						
x_7	48						

Another Example



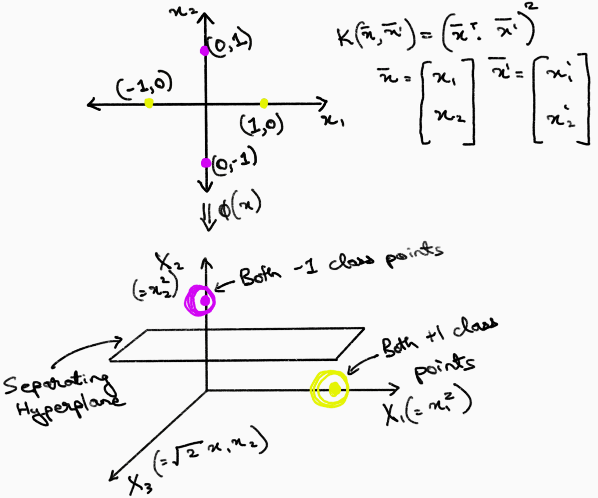
Q) What is $\phi(x)$?

$$K(\bar{x}, \bar{x'}) = \phi(\bar{x})\phi(\bar{x'})$$

$$K(\bar{x}, \bar{x'}) = \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \right\}^2 = (x_1x_1' + x_2x_2')^2$$

$$\implies \phi(x) =< x_1^2, \sqrt{2}x_1x_2, x_2^2 >= x_1^2x_1'^2 + x_2^2x_2'^2 + 2x_1x_1'x_2x_2'$$

Another Example



Some Kernels

1. Linear: $K(\bar{x}_1, \bar{x}_2) = \bar{x}_1\bar{x}_2$
2. Polynomial: $K(\bar{x}_1, \bar{x}_2) = (p + \bar{x}_1\bar{x}_2)^q$
3. Gaussian: $K(\bar{x}_1, \bar{x}_2) = e^{-\gamma||\bar{x}_1 - \bar{x}_2||^2}$ where $\gamma = \frac{1}{2\sigma^2}$ - Also called Radial Basis Function (RBF)

Q) For $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ what space does kernel $\mathrm{K}(\bar{x}, \bar{x'}) = (1 + \bar{x}\bar{x'})^3$ belong to?

$\bar{x} \in \mathbb{R}^2$

$\phi(\bar{x}) \in \mathbb{R}^?$

$$
\begin{aligned}
K(x, z) &= (1 + x_1 z_1 + x_2 z_2)^3 \\
&= \dots \\
&= < 1, x_1, x_2, x_1^2, x_2^2, x_1^2 x_2, x_1 x_2^2, x_1^3, x_2^3, x_1 x_2 >
\end{aligned}
$$

10 dimensional?

Kernels

Q) For $\bar{x} = x$; what space does RBF kernel lie in?

$$K(x, z) = e^{-\gamma||x-z||^2}$$
$$= e^{-\gamma(x-z)^2}$$

Now:

$$e^{\alpha} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}$$

$\therefore e^{-\gamma(x-z)^2}$ is ∞ dimensional!!

SVM: Parametric or Non-Parametric

Q) Is SVM parametric or non-parametric?

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Q) Is SVM parametric or non-parametric?

Yes and No

Yes $\rightarrow$ Linear kernel or polynomial kernel (form fixed)

No $\rightarrow$ RBF (form changes with data)

RBF is Non-Parametric

$$\hat{y}(x_{test}) = sign(\bar{w}\bar{x}_{test} + b)$$

$$= sign(\sum_{j=1}^{N_{SV}} \alpha_j y_j \bar{x}_j \bar{x}_{test} + b)$$

$$\hat{y}(X_{test}) = sign(\sum_{j=1}^{N} \alpha_j y_j K(\bar{x}_j, \bar{x}_{test}) + b)$$

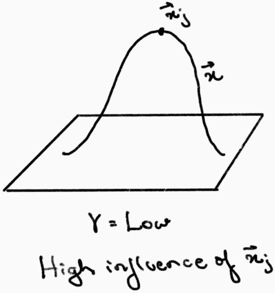
$\alpha_j = 0$ where $j \neq$ S.V.

RBF is Non-Parametric

Now $K(\bar{x}_j, \bar{x}_{test})$ for RBF is:

$$e^{-\gamma||\bar{x}_j - \bar{x}_{test}||^2}$$

$\therefore$ Hypothesis is a function of "all" train points
Closer $\bar{x}$ is to $\bar{x}_N$; more is it influencing $\hat{y}(\bar{x})$ - hypothesis function



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- Now if we add a point to the dataset
- Functional form can adapt (similar to KNN)
- $\therefore$ SVM with RBF kernel is non-parametric

Interpretation of RBF

- $\hat{y}(x) = sign(\sum \alpha_i y_i e^{-||x-x_i||^2} + b)$

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- $\hat{y}(x) = sign(\sum \alpha_i y_i e^{-||x-x_i||^2} + b)$
- $-||x - x_i||^2$ corresponds to radial term
- $\sum \alpha_i y_i$ is the activation component
- $e^{-||x-x_i||^2}$ is the basis component

γ: How far is the influence of a single training sample