Geometric Interpretation of Linear Regression

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Linear Combination of Vectors

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$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_i v_i$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_i \in \mathbb{R}$

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It is the set of all vectors that can be generated by linear combinations of $v_1, v_2, \ldots, v_i$.

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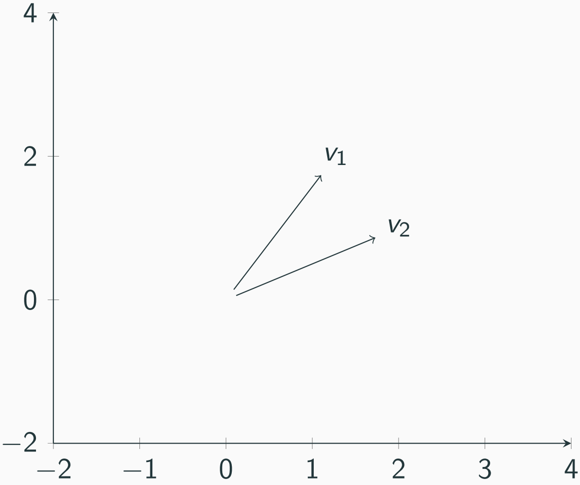

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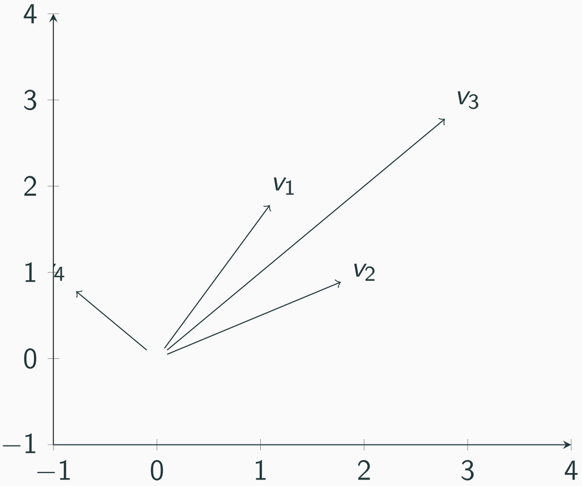

$v_3 = v_1 + v_2$ and $v_4 = v_1 - v_2$

$\text{Span}((v_1, v_2)) \in \mathcal{R}^2$

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No

Span of the above set is along the line $y = 2x$

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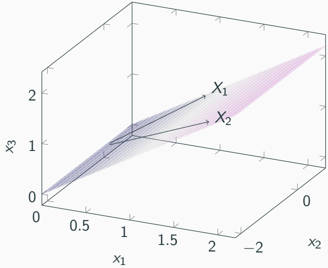

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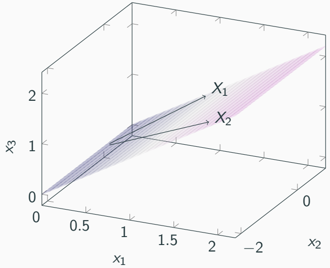

The span is the plane $z = x$ or $x_3 = x_1$

Geometric Interpretation

Consider X and y as follows.

$$\mathbf{X} = \begin{pmatrix} 1 & 2 \\ 1 & -2 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 8.8957 \\ 0.6130 \\ 1.7761 \end{pmatrix}$$

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- We wish to find $\hat{y}$ such that

$$\underset{\hat{y} \in SPAN\{\bar{x_1}, \bar{x_2}, \ldots, \bar{x_D}\}}{\arg\min} ||y - \hat{y}||_2$$

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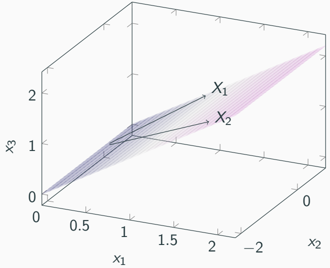

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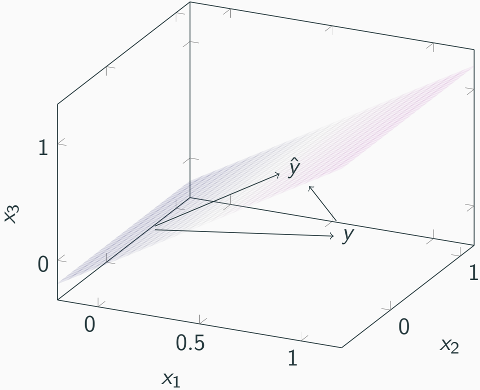

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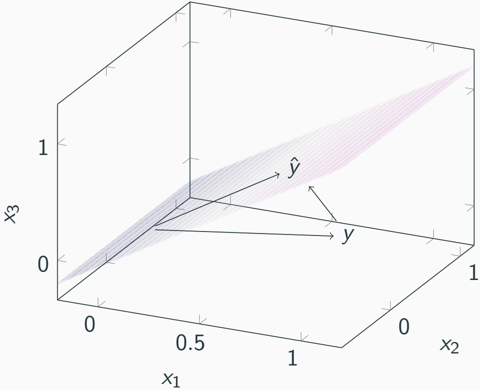

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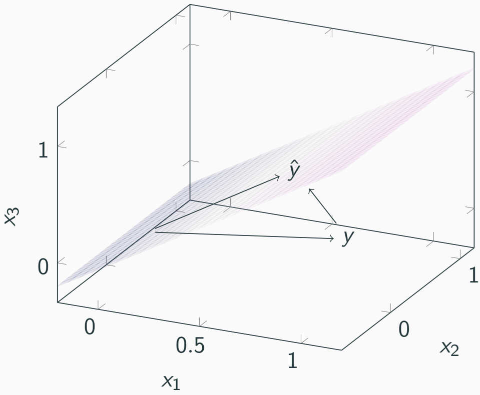

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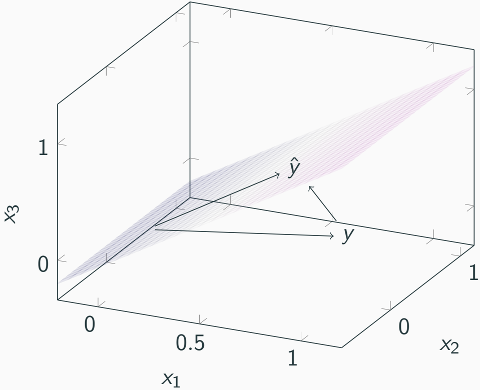

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- This happens when $y - \hat{y} \perp x_j \forall j$ or $x_j^T(y - \hat{y}) = 0$
- $X^T(y - X\theta) = 0$
- $X^T y = X^T X\theta$ or $\hat{\theta} = (X^T X)^{-1} X^T y$