HIDDEN MARKOV MODEL

- NIPUN BATRA

SOME APPLICATIONS

1. ACTIVITY MONITORING

SOME APPLICATIONS

1 ACTIVITY MONITORING

Accelerometer

Activity WALKING RUNNING SITTING

SOME APPLICATIONS

① ACTIVITY MONITORING

Accelerometer 

Activity WALKING RUNNING SITTING

② SPECH RECOGNITION

RAW SPECH 

SOME APPLICATIONS

1. ACTIVITY MONITORING

Accelerometer

Activity WALKING RUNNING SITTING

2. SPECH RECOGNITION

RAW SPECH

FEATURES F_1, F_2 F_1, F_2 . . — — . . .

WORDS I LIKE MACHINE LEARNING...

SOME APPLICATIONS

1. ACTIVITY MONITORING
2. SPECH RECOGNITION
3. PART OF SPEECH TAGGING
4. GENE FINDING

⋮

SEQUENTIAL DATA

1) Amount of rainfall daily

2) Price of stock

3) Words in a sentence

4) Energy consumption of a home

WEATHER EXAMPLE

Let $x_t \in \begin{cases} R: \text{Rainy} \\ S: \text{Sunny} \\ C: \text{Cloudy} \end{cases}$

be observed outlook at 't^{th}' day

Sample observations: R R R C C R R S S S C S S

Modelling Sequential Data as i.i.d

* Easiest way to treat sequential data ⟶ ignore sequential aspect and treat observations as i.i.d.

DOWNSIDE?

Modelling Sequential Data as i.i.d

* Easiest way to treat sequential data → ignore sequential aspect and treat observations as i.i.d.

DOWNSIDE?

* Fails to exploit sequential patterns
  → Nearby observations "correlated"

Modelling Sequential Data as i.i.d

* Easiest way to treat sequential data $\longrightarrow$ ignore sequential aspect and treat observations as i.i.d.

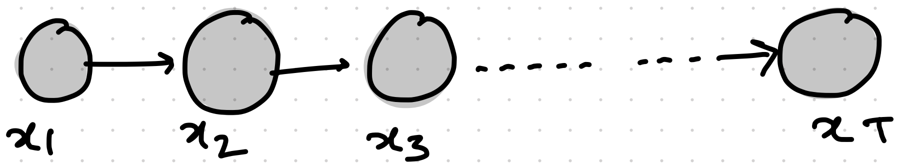
DOWNSIDE?

* Fails to exploit sequential patterns
  $\longrightarrow$ Nearby observations "correlated"

* Q: Predict if it will rain today or not?

  $\because$ Rain today $\xrightarrow[\text{w/ high probability}]{}$ Rain tomorrow

MARKOV MODEL



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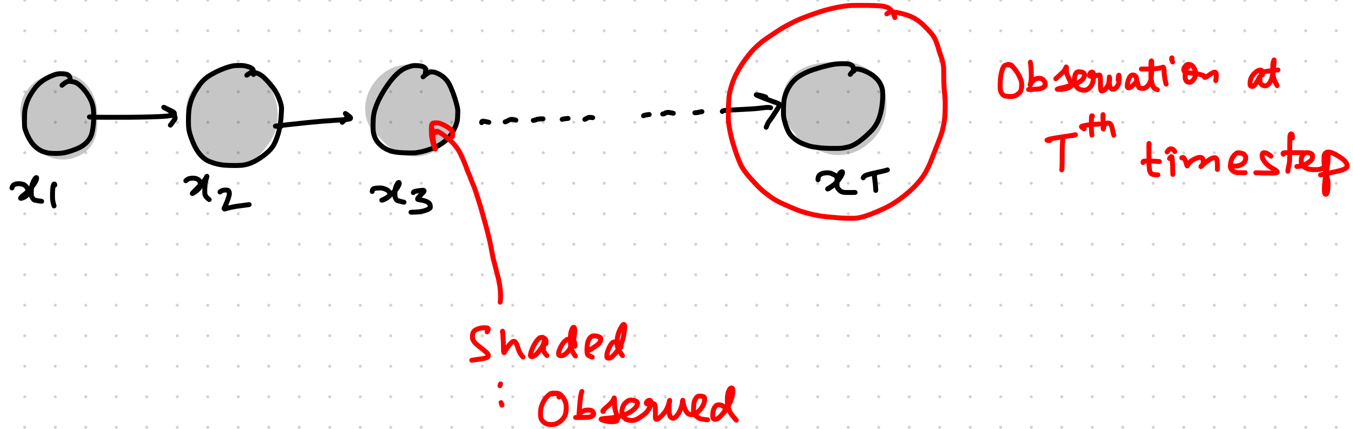

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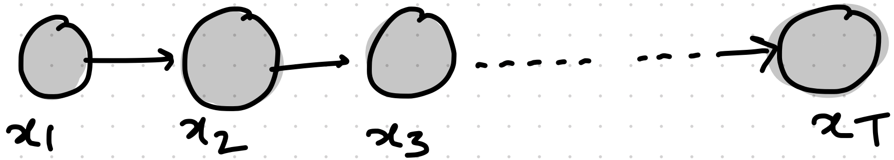
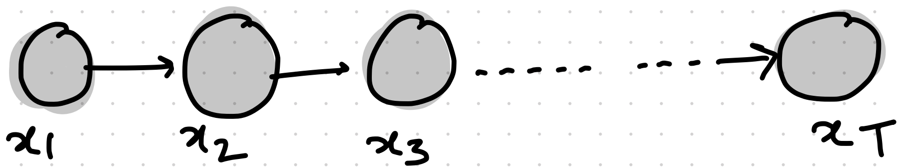


FIRST ORDER MARKOV CHAIN

Future prediction independent of past given present

MARKOV MODEL

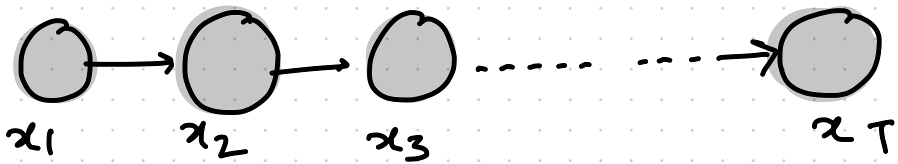


FIRST ORDER MARKOV CHAIN

Future prediction independent of past given present

$$\Rightarrow \quad p(x_{t+1} \mid x_1, x_2 \dots x_t) = p(x_{t+1} \mid x_t)$$

MARKOV MODEL



FIRST ORDER MARKOV CHAIN

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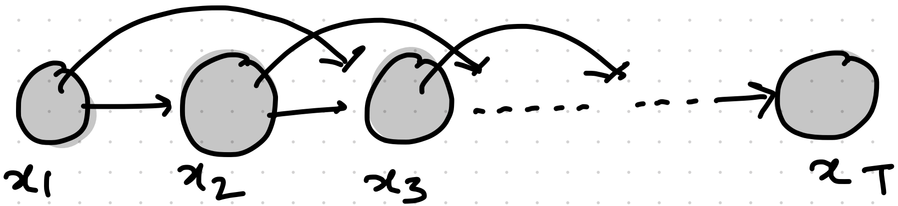
JOINT PROB $= p(x_1, \ldots, x_T) = p(x_1) \cdot p(x_2 \mid x_1) \cdot p(x_3 \mid x_2) \cdots p(x_T \mid x_{T-1})$

MARKOV MODEL

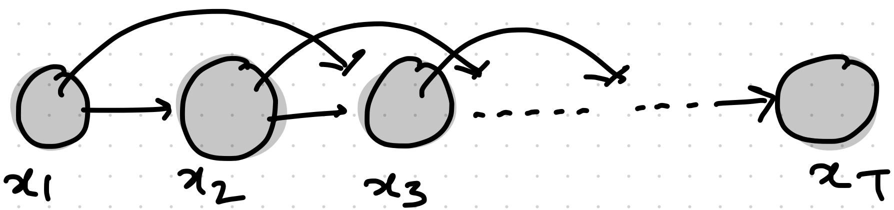


SECOND ORDER MARKOV CHAIN

$$p(x_{t+1} \mid x_1, x_2 \ldots x_t) = p(x_{t+1} \mid x_t, x_{t-1})$$

JOINT PROB $= p(x_1, \ldots, x_T) =$

MARKOV MODEL



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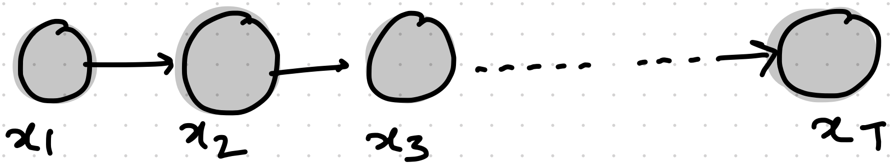
JOINT PROB $= p(x_1, \ldots, x_T) = p(x_1) \cdot p(x_2 \mid x_1) \cdot \prod_{n=3}^{T} p(x_n \mid x_{n-1}, x_{n-2})$

PARAMETERS OF FIRST ORDER MARKOV MODEL



FIRST ORDER MARKOV CHAIN

Future prediction independent of past given present

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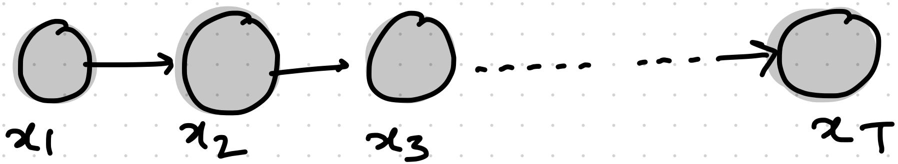
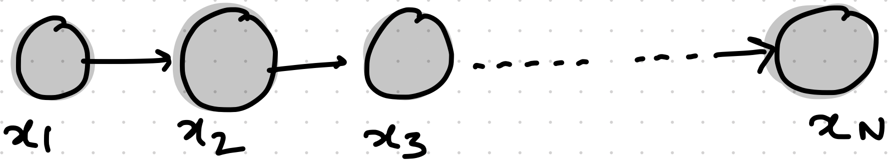
PARAMETERS OF FIRST ORDER MARKOV MODEL



① MAIN ASSUMPTION : STATIONARITY

Data evolves over time, but distribution from which data is generated is fixed.

PARAMETERS OF FIRST ORDER MARKOV MODEL

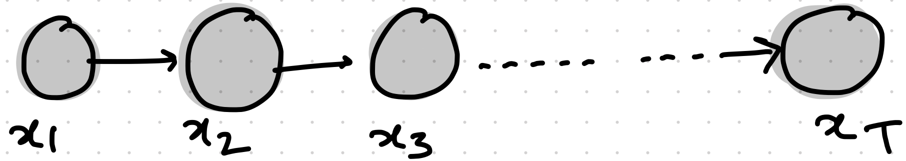


① MAIN ASSUMPTION : STATIONARITY

Data evolves over time, but distribution from which data is generated is fixed.

② We would like to have parameter sharing (parameters independent of time)

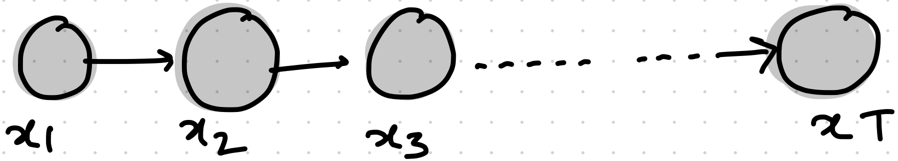
PARAMETERS OF FIRST ORDER MARKOV MODEL



we would like to have parameter sharing

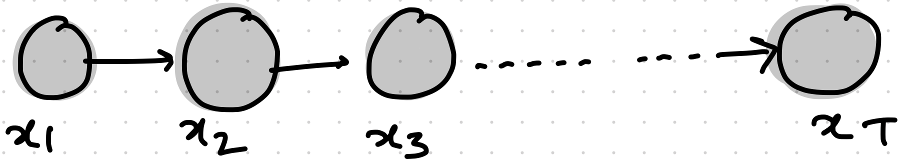
$$p(x_t \mid x_{t-1}) \text{ is same } \forall\, t$$

PARAMETERS OF FIRST ORDER MARKOV MODEL



$$p(x_1, x_2 \ldots x_T) = p(x_1) \prod_{t=2}^{T} p(x_t | x_{t-1})$$

PARAMETERS OF FIRST ORDER MARKOV MODEL



$$p(x_1, x_2 \dots x_T) = p(x_1) \prod_{t=2}^{T} p(x_t | x_{t-1})$$

$$\text{Parameters} = \theta = \{ \pi , A \}$$

Prior probability

Transition matrix

PARAMETERS OF FIRST ORDER MARKOV MODEL

TRANSITION MATRIX (A)

ASSUME x_t can take 1 of K states

$$A_{jk} \equiv p(x_t = k \mid x_{t-1} = j)$$

'j' and 'k' $\in \{1, \ldots K\}$

PARAMETERS OF FIRST ORDER MARKOV MODEL

ASSUME x_t can take 1 of K states

$$A_{jk} \equiv p(x_t = k \mid x_{t-1} = j)$$

'j' and 'k' $\in \{1, \dots K\}$

$j \longrightarrow k$

PARAMETERS OF FIRST ORDER MARKOV MODEL

Representing $A_{jk} \equiv p(x_t = k \mid x_{t-1} = j)$

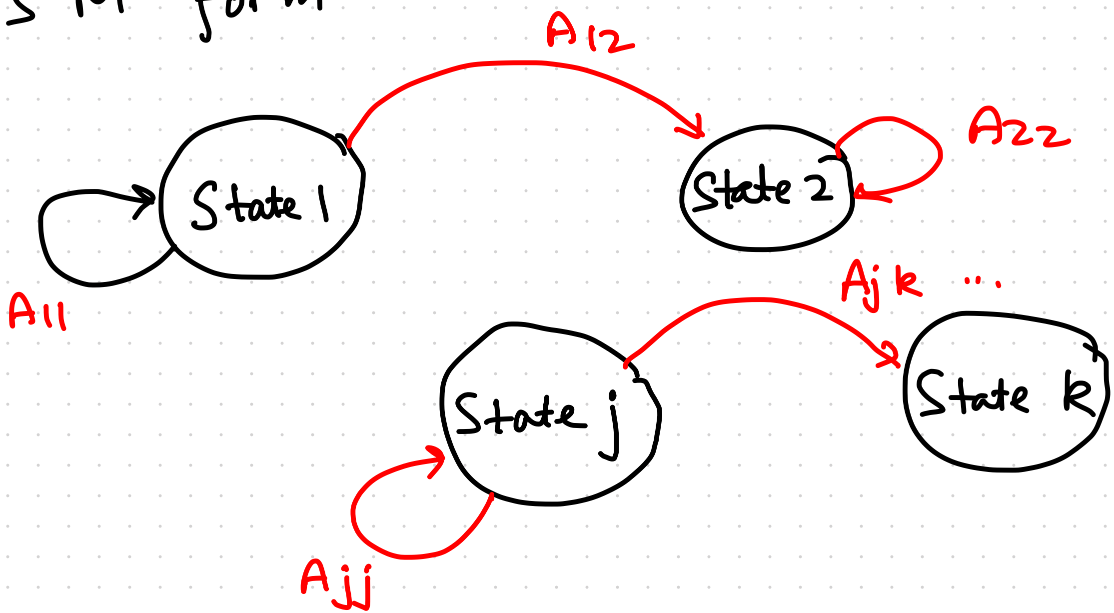
① Tabular form

x_{t-1}	x_t	$p(x_t \mid x_{t-1})$
1	1	A_{11}
2	1	A_{21}
$\vdots$		$\cdot$
j	k	A_{jk}
K	K	A_{KK}

PARAMETERS OF FIRST ORDER MARKOV MODEL

Representing $A_{jk} \equiv p(x_t = k \mid x_{t-1} = j)$

② FSM form



PARAMETERS OF FIRST ORDER MARKOV MODEL

Representing $A_{jk} \equiv p(x_t = k \mid x_{t-1} = j)$

③ Adjacency Matrix

$$
\begin{array}{c|ccccc}
x_{t-1} \backslash x_t & 1 & 2 & \cdots & k & K \\
\hline
1 & & & & & \\
2 & & & & & \\
\vdots & & & & & \\
j & & & & A_{jk} & \\
\end{array}
$$

PARAMETERS OF FIRST ORDER MARKOV MODEL

Prior Probability (π)

$$\pi_k = p(z_1 = k)$$

Probability of 1st observation

Q) Given A & π as:

$$A = \begin{array}{c} \\ R \\ C \\ S \end{array} \begin{array}{c} \begin{array}{ccc} R & C & S \end{array} \\ \begin{bmatrix} .4 & .3 & .3 \\ .2 & .6 & .2 \\ .1 & .1 & .8 \end{bmatrix} \end{array}$$

$$\pi = \begin{array}{c} \begin{bmatrix} .2 & .4 & .4 \end{bmatrix} \\ \begin{array}{ccc} R & C & S \end{array} \end{array}$$

WHAT IS PROBALITY OF SEQUENCE

$$x = \{R\ R\ S\ S\ C\}$$

Q) Given A & π as:

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$$\pi = \begin{array}{ccc} .2 & .4 & .4 \\ R & C & S \end{array}$$

WHAT IS PROBALITY OF SEQUENCE

$$x = \{R\ R\ S\ S\ C\}$$

$$P(x \mid \theta = \{A, \pi\}) = P(x_1 = R) \cdot P(x_2 = R \mid x_1 = R) \cdot P(x_3 = S \mid x_2 = R) \cdot P(x_4 = S \mid x_3 = S) \cdot P(x_5 = C \mid x_4 = S)$$

$$= \pi_R \cdot A_{RR} \cdot A_{RS} \cdot A_{SS} \cdot A_{SC}$$

Q) Given A & π as:

$$A = \begin{array}{c} \\ R \\ C \\ S \end{array} \begin{array}{c} \begin{array}{ccc} R & C & S \end{array} \\ \begin{bmatrix} .4 & .3 & .3 \\ .2 & .6 & .2 \\ .1 & .1 & .8 \end{bmatrix} \end{array}$$

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$p(x_1 = R) = 0.2$. what is $p(x_2 = R)$?

Q) Given A & π as:

$$A = \begin{array}{c|ccc} & R & C & S \\ \hline R & .4 & .3 & .3 \\ C & .2 & .6 & .2 \\ S & .1 & .1 & .8 \end{array} \qquad \pi = \begin{array}{ccc} R & C & S \\ \hline .2 & .4 & .4 \end{array}$$

$p(x_1 = R) = 0.2$. what is $p(x_2 = R)$?

$$p(x_2 = R) = p(x_1 = R) . p(x_2 = R \mid x_1 = R) + p(x_1 = S) . p(x_2 = R \mid x_1 = S) + p(x_1 = C) . p(x_2 = R \mid x_1 = C)$$

Q) Given A & π as:

$$A = \begin{array}{c c} & \begin{array}{ccc} R & C & S \end{array} \\ \begin{array}{c} R \\ C \\ S \end{array} & \begin{bmatrix} .4 & .3 & .3 \\ .2 & .6 & .2 \\ .1 & .1 & .8 \end{bmatrix} \end{array} \qquad \pi = \begin{array}{c} \begin{bmatrix} .2 & .4 & .4 \end{bmatrix} \\ \begin{array}{ccc} R & C & S \end{array} \end{array}$$

$p(x_1 = R) = 0.2$. what is $p(x_2 = R)$?

$$p(x_2 = R) = p(x_1 = R) . p(x_2 = R \mid x_1 = R) +$$
$$p(x_1 = S) . p(x_2 = R \mid x_1 = S) +$$
$$p(x_1 = C) . p(x_2 = R \mid x_1 = C)$$

$$= (.2) * (.4) + (.4) * (0.1) + (.4) * (0.2) = 0.2 = p(x_1 = R)$$

Hidden Markov Model

what is the hidden component

Hidden Markov Model

what is the hidden component

Example: Coin Toss Model

Assume 2 coins
- Biased ($p(H) = 0.7$)
- Unbiased / Fair ($p(H) = 0.5$)

Hidden Markov Model

What is the hidden component

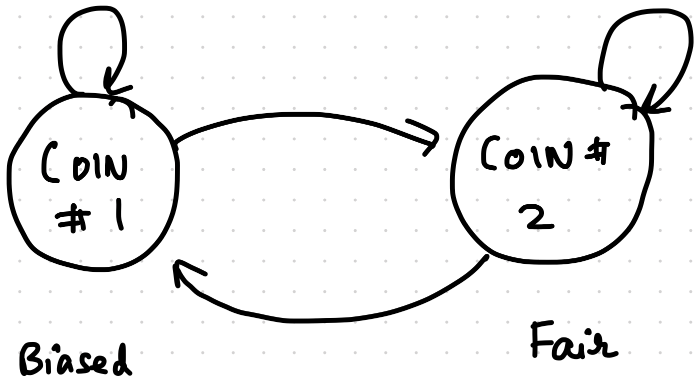
Example: Coin Toss Model

Assume 2 coins → Biased $(p(H) = 0.7)$
→ Unbiased / Fair $(p(H) = 0.5)$

We see only sequence of observations $x = \{H, T \ldots\}$

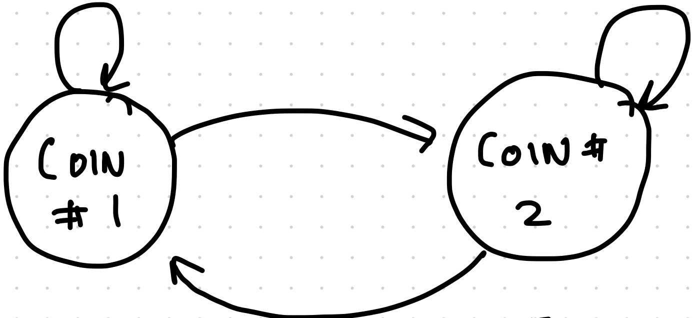
Hidden: which coin was tossed!

HMM (example)



Markov Model

HMM (example)



Markov Model

Biased

Fair

Emission

$p(H) = 0.7$
$p(T) = 0.3$

$p(H) = 0.5$
$p(T) = 0.5$

HMM (example)

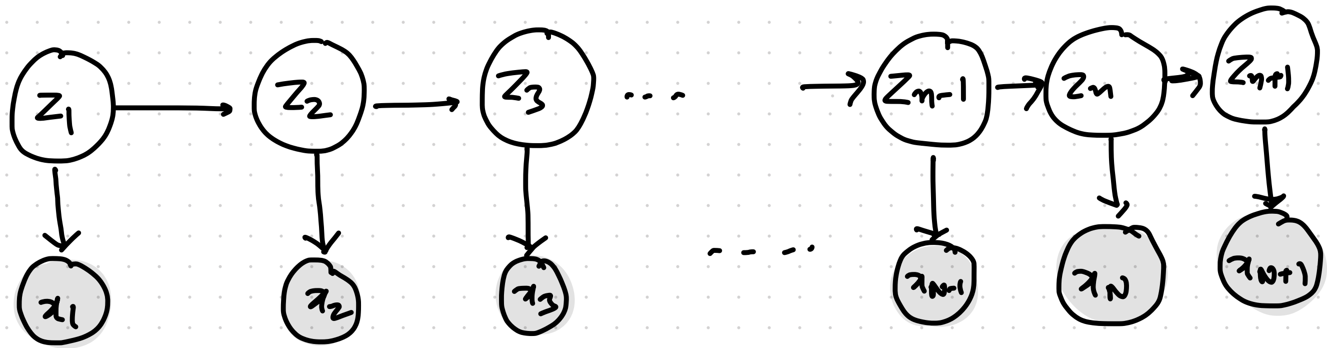
$z_i = \{\text{coin } 1, \text{coin } 2\}$

$$z_1 \to z_2 \to z_3 \cdots \to z_{n-1} \to z_n \to z_{n+1}$$

HMM (example)

$Z_i = \{\text{coin 1}, \text{coin 2}\}$

$x_i = \{H, T\}$



HMM (example)

$z_i = \{\text{coin 1}, \text{coin 2}\}$

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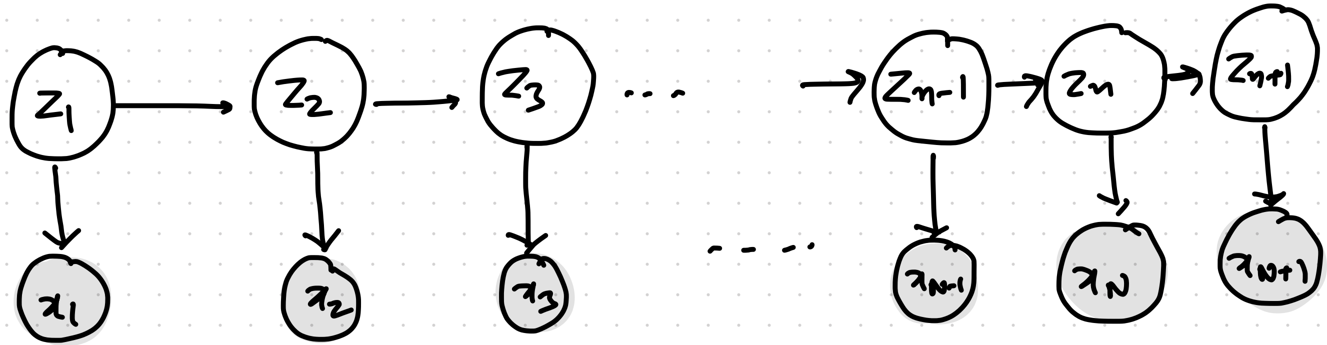

x_i : Observed

z_i : Latent state

HMM Parameters

1. Transition matrix A:

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HMM Parameters

① Transition matrix A:

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$$\pi_k = p(z_1 = k)$$

HMM Parameters

1. Transition matrix A:

$$A_{jk} = p(z_n = k \mid z_{n-1} = j)$$

2. Prior prob. π:

$$\pi_k = p(z_1 = k)$$

3. Emission Probability : ϕ define $p(x_n \mid z_n, \phi)$

HMM Parameters

(3) Emission Probability : ϕ define $p(x_n | z_n, \phi)$

Case I : ϕ is discrete.

e.g. $z_i = \{\text{Fair}, \text{Biased}\}$

ϕ defined in terms of conditional probability

Fair	Biased
$P(H) = 0.5$	$P(H) = 0.7$
$P(T) = 0.5$	$P(T) = 0.3$

HMM Parameters

③ Emission Probability : ϕ define $p(x_n | z_n, \phi)$

<u>Case II</u> ϕ is continuous

State: Appliance is ON / OFF

ϕ : ON : Power $\sim N(100, 10)$

OFF : Power $\sim N(0, 5)$

SOME APPLICATIONS

(1) ACTIVITY MONITORING

Accelerometer

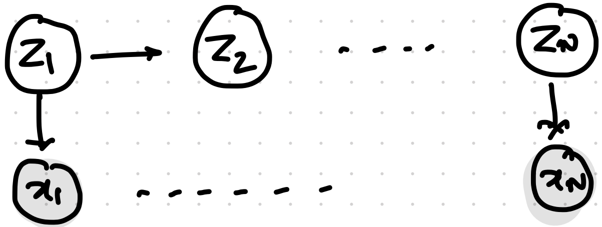
Activity

WALKING z_1 | RUNNING z_2 | SITTING z_3

$z_i \in \{$ WALKING, RUNNING, SITTING $\}$

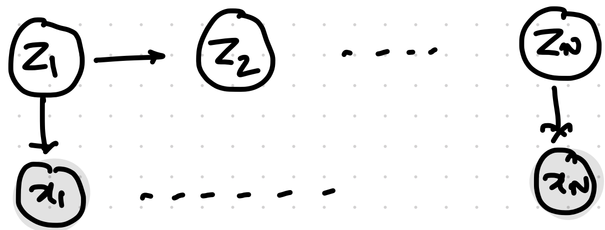
$x_i \in$ (SOME TRANSFORMATION OF RAW DATA)

PI: HMM sampling



Given π, A, ϕ generate $X = \{x_1 \dots x_N\}$ & $Z = \{z_1 \dots z_N\}$

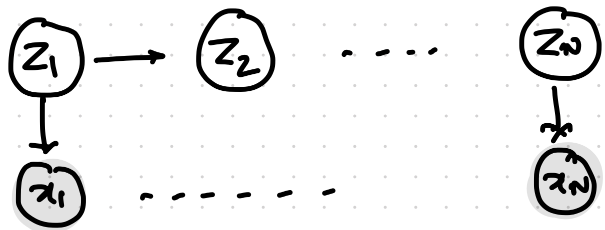
PI: HMM sampling



Given π, A, ϕ generate $X = \{x_1 \ldots x_N)$ & $Z = \{z_1 \ldots z_N\}$

① CHOOSE z_1 as per π

PI: HMM sampling

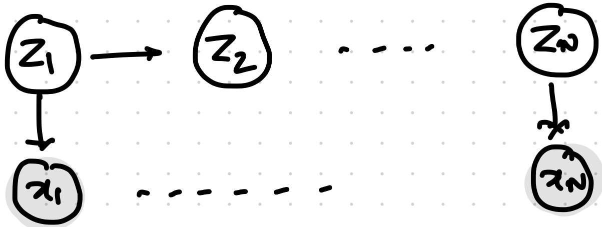


Given π, A, ϕ generate $X = \{x_1 \ldots x_N\}$ & $Z = \{z_1 \ldots z_N\}$

① CHOOSE z_1 as per π

② Sample x_1 using ϕ and z_1

PI: HMM sampling



Given π, A, ϕ generate $X = \{x_1 \ldots x_N)$ & $Z = \{z_1 \ldots z_N\}$

① CHOOSE z_1 as per π

② Sample x_1 using ϕ and z_1

③ For $n = 2 : N$

(3.1) Sample z_n from z_{n-1} using A and z_{n-1}

(3.2) Sample x_n from z_n using ϕ and z_n

PI: HMM sampling

Notebooks: Discrete HMM Simulatⁿ (Unfair Casino)

Continuous HMM Simulatⁿ (Power Appliance)

P II HMM Evidence likelihood

$X = \{x_1 \dots x_T\}$

Given X, $\underbrace{\pi, A, \phi}_{\theta}$ what is $L(X|\theta)$?

PII: HMM Evidence likelihood

$X = \{x_1 \dots x_T\}$

Given X, $\underbrace{\pi, A, \phi}_{\theta}$ what is $L(X|\theta)$?

Likelihood $= p(X|\theta) = \sum_{z} p(X, z|\theta)$

(Marginalisat^n)

FOR $X = \{H, H\}$ what is $L(X|\theta)$ where $\theta = ?$

$$\pi = \begin{array}{cc} [\,0.6 & 0.4\,] \\ \text{Bias} & \text{Fair} \end{array}$$

$$A = \begin{array}{ccc} & B & F \\ \text{Bias} & 0.9 & 0.1 \\ \text{Fair} & 0.1 & 0.9 \end{array}$$

$\phi =$

Bias

$p(H) = 0.7$

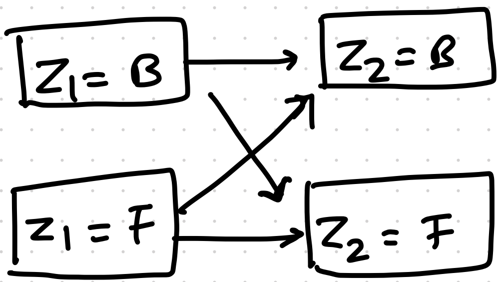
$p(T) = 0.3$

Fair

$p(H) = 0.5$

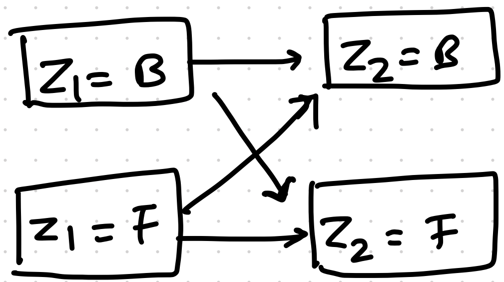
$p(T) = 0.5$

FOR $X = \{H, H\}$ what is $L(X|\theta)$ where $\theta = ?$



Trellis Diagram
for 2 states
and 2 timesteps

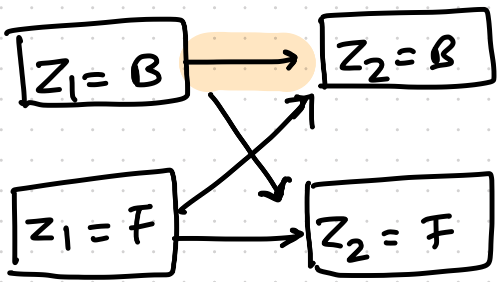
FOR $X = \{H, H\}$ what is $L(X|\theta)$ where $\theta =$



Trellis Diagram for 2 states and 2 timesteps

$$L(X|\theta) = p(z_1 = B) \cdot p(x_1 = H | z = B) \cdot p(z_2 = B | z_1 = B) \cdot p(x_2 = H | z = B)$$

FOR $X = \{H, H\}$ what is $L(X|\theta)$ where $\theta =$



Trellis Diagram
for 2 states
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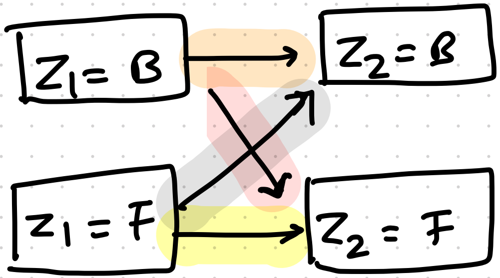
$$L(X|\theta) = p(z_1 = B) \cdot p(x_1 = H \mid z = B) \cdot p(z_2 = B \mid z_1 = B) \cdot p(x_2 = H \mid z = B)$$

FOR $X = \{H, H\}$ what is $L(X|\theta)$ where $\theta = ?$



Trellis Diagram for 2 states and 2 timesteps

$$L(X|\theta) = p(z_1 = B) \cdot p(x_1 = H | z = B) \cdot p(z_2 = B | z_1 = B) \cdot p(x_2 = H | z = B)$$

$$+ \; p(z_1 = F) \cdot p(x_1 = H | z = F) \cdot p(z_2 = B | z_1 = F) \cdot p(x_2 = H | z = B)$$

$$+ \; p(z_1 = B) \cdot p(x_1 = H | z = B) \cdot p(z_2 = F | z_1 = B) \cdot p(x_2 = H | z = F)$$

$$+ \; p(z_1 = F) \cdot p(x_1 = H | z = F) \cdot p(z_2 = F | z_1 = F) \cdot p(x_2 = H | z = F)$$

FOR $X = \{H, H\}$ what is $L(X|\theta)$ where $\theta = 2$

$$L(X|\theta) = p(z_1 = B) \cdot p(x_1 = H | z = B) \cdot p(z_2 = B | z_1 = B) \cdot p(x_2 = H | z = B)$$

$$+ \; p(z_1 = F) \cdot p(x_1 = H | z = F) \cdot p(z_2 = B | z_1 = F) \cdot p(x_2 = H | z = B)$$

$$+ \; p(z_1 = B) \cdot p(x_1 = H | z = B) \cdot p(z_2 = F | z_1 = B) \cdot p(x_2 = H | z = F)$$

$$+ \; p(z_1 = F) \cdot p(x_1 = H | z = F) \cdot p(z_2 = F | z_1 = F) \cdot p(x_2 = H | z = F)$$

$$= (0.6)(0.7)(0.9)(0.7) \qquad = 0.2646 \qquad = 0.3896$$

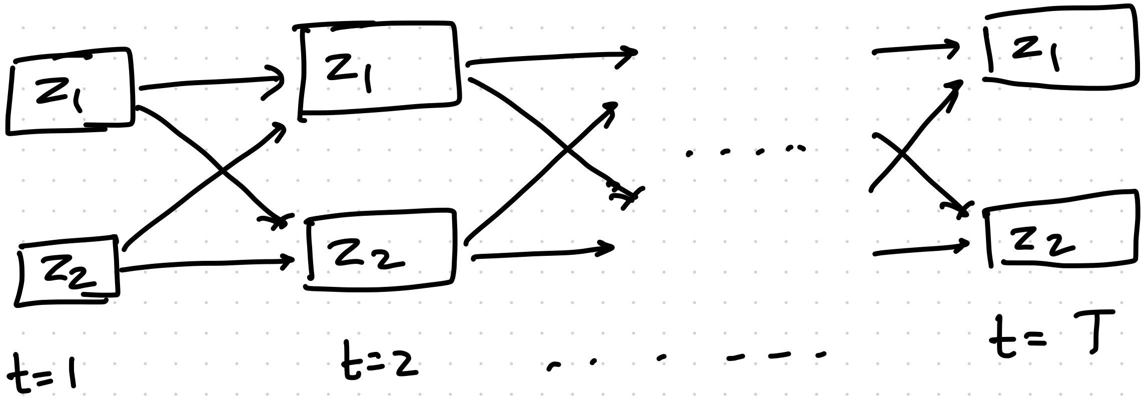
$$+ (0.4)(0.5)(0.1)(0.7) \qquad + 0.0140$$

$$+ (0.6)(0.7)(0.1)(0.5) \qquad + 0.0210$$

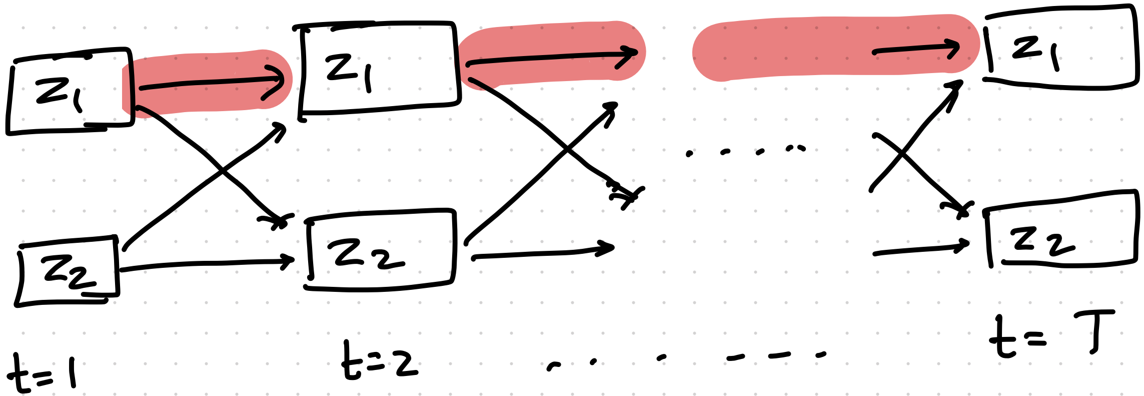
$$+ (0.4)(0.5)(0.9)(0.5) \qquad + 0.0900$$

FOR $X = \{\underbrace{H, H, \ldots}_{T \text{ timesteps.}}\}$ what is $L(X|\theta)$ where $\theta = 1$

FOR $X = \{\underbrace{H, H, \ldots}_{T \text{ timesteps.}}\}$ what is $L(X|\theta)$ where $\theta =$?



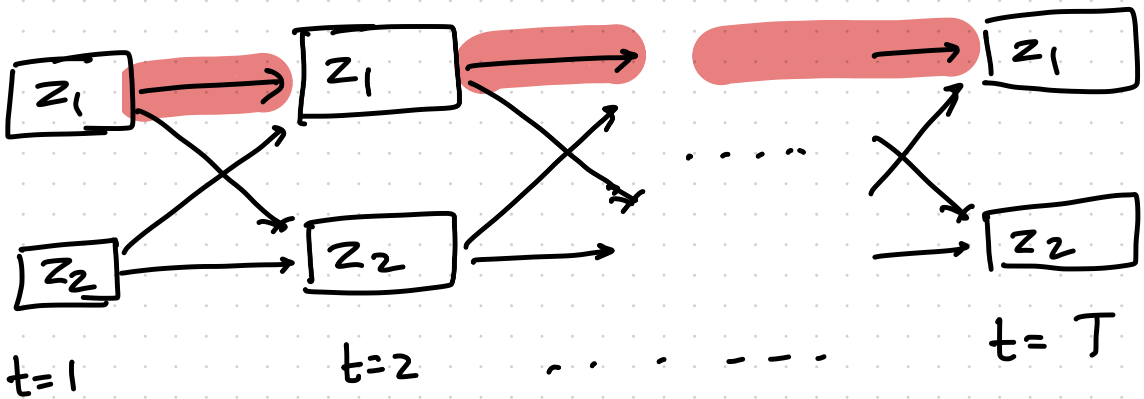
FOR $X = \{\underbrace{H, H, \ldots}_{T \text{ timesteps}}\}$ what is $L(X|\theta)$ where $\theta = ?$



SNO	1	2	TIME	T
1	z_1	z_1		z_1
⋮				
K^T	...	...		..

FOR $X = \{H, H, \ldots\}$ what is $L(X|\theta)$ where $\theta = ?$

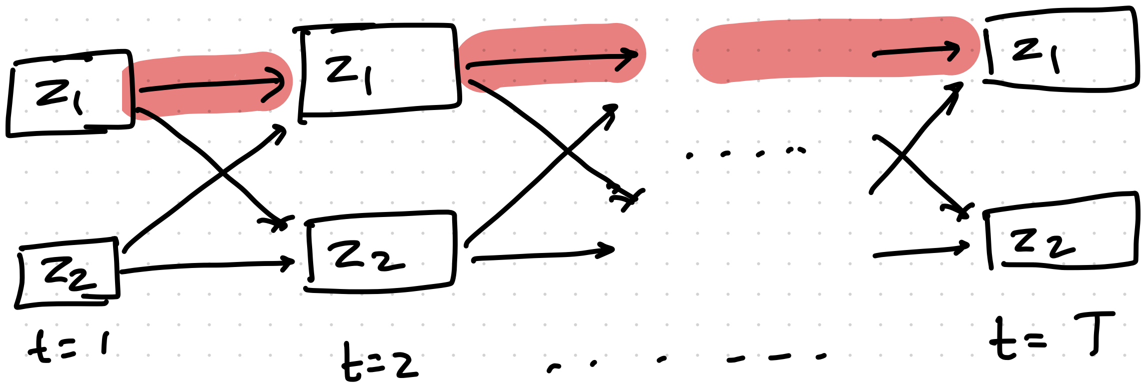
$\underline{\hphantom{H, H, \ldots}}$ T timesteps.



SNO	TIME 1	2	...	T
1	z_1	z_1		z_1
⋮				
K^T	...	...		..

Total K^T paths required for $L(X|\theta)$

FOR $X = \underbrace{\{H, H, \ldots\}}_{T \text{ timesteps.}}$ what is $L(X|\theta)$ where $\theta = ;$



Length of each path $= T$

$\Rightarrow O(T)$ multiplications

Total K^T paths required for $L(X|\theta)$

$P(X|\theta)$ computat^n is $O(TK^T)$

Can we do better? Hint: Dynamic Programming

FORWARD PROCEDURE

Define $\alpha_t(i) = P(x_{1:t}, z_t = i)$

FORWARD PROCEDURE

Define $\alpha_t(i) = P(x_{1:t}, z_t = i)$

i.e. $\alpha_t(i)$ = Probability of being in state 'i' at time t and observatⁿ $x_{1:t}$

FORWARD PROCEDURE (Base Step $t=1$)

Define $\alpha_t(i) = P(x_{1:t}, z_t = i)$

For our example for $t=1$

$$\alpha_1(B) = P(x_1, z_1 = \text{Biased}) = \pi_B \cdot P(x_1 = H \mid z_1 = B)$$

$$= 0.6 * 0.7 = 0.42$$

FORWARD PROCEDURE (Base Step $t=1$)

Define $\alpha_t(i) = P(x_{1:t}, z_t = i)$

For our example for $t=1$

$$\alpha_1(\text{Bias}) = P(x_1, z_1 = \text{Biased}) = \pi_B \cdot P(x_1 = H \mid z_1 = B)$$

$$= 0.6 * 0.7 = 0.42$$

$$\alpha_1(\text{Fair}) = P(x_1 = H, z_1 = F) = \pi_F \cdot P(x_1 = H \mid z_1 = F)$$

$$= 0.4 * 0.5 = 0.20$$

FORWARD PROCEDURE (INDUCTION STEP)

Define $\alpha_t(i) = P(x_{1:t}, Z_t = i)$

$\alpha_1 = 0.42$

$\boxed{Z_B}$

$\boxed{Z_F}$

$\alpha_1 = 0.2$

FORWARD PROCEDURE (INDUCTION STEP)

Define $\alpha_t(i) = P(x_{1:t}, z_t = i)$

$\alpha_1 = 0.42$ $\qquad\qquad$ $\alpha_2(B) = ?$

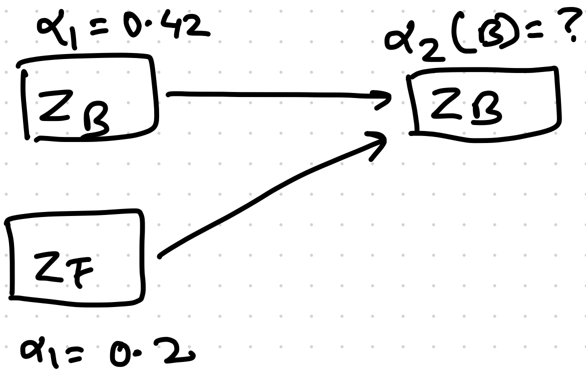
$\boxed{Z_B}$ $\qquad\qquad$ $\boxed{Z_B}$

$\boxed{Z_F}$

$\alpha_1 = 0.2$

FORWARD PROCEDURE (INDUCTION STEP)

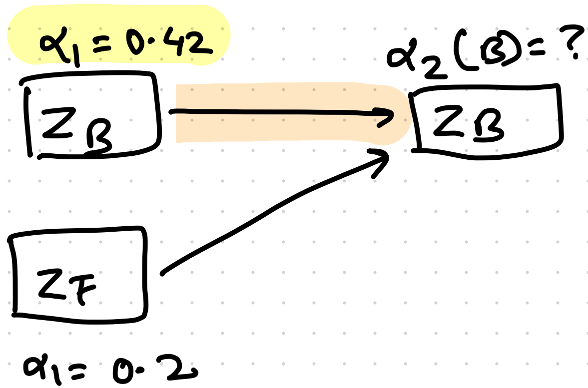
Define $\alpha_t(i) = P(x_{1:t}, z_t = i)$



2 paths (K) to Bias state at time step 2

FORWARD PROCEDURE (INDUCTION STEP)

Define $\alpha_t(i) = P(x_{1:t}, z_t = i)$

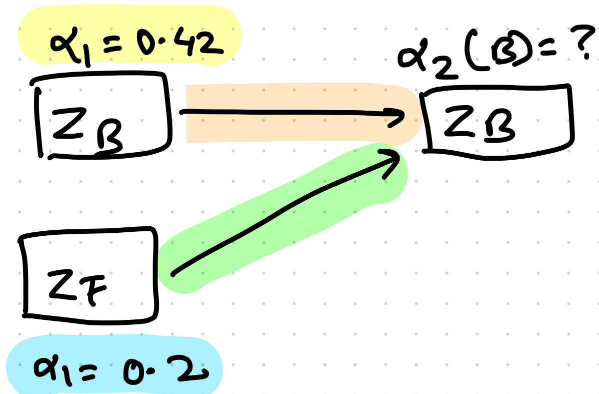


2 paths (K) to Bias state at time step 2

$$P(x_1 = H, x_2 = H, z_2 = B) = \left[P(x_1 = H, z_1 = B) \cdot A_{BB}\right] P(H \mid Bias)$$

FORWARD PROCEDURE (INDUCTION STEP)

Define $\alpha_t(i) = P(x_{1:t}, z_t = i)$



2 paths (K) to Bias state at time step 2

$$P(x_1 = H, x_2 = H, z_2 = B) = [P(x_1 = H, z_1 = B) \cdot A_{BB}] \, P(H \mid Bias)$$

$$+ [P(x_1 = H, z_1 = F) \cdot A_{FB}] \, P(H \mid B)$$

FORWARD PROCEDURE (INDUCTION STEP)

Define $\alpha_t(i) = P(x_{1:t}, z_t = i)$

$$P(x_1 = H, x_2 = H, z_2 = B) = [P(x_1 = H, z_1 = B) . A_{BB}] P(H|Bias)$$
$$+ [P(x_1 = H, z_1 = F) . A_{FB}] P(H|B)$$

$$\Rightarrow \alpha_2(B) = (\alpha_1(B) A_{BB}) \phi_B(x=H) + (\alpha_1(F) A_{FB}) \phi_B(x=H)$$

$$= (0.42) * (0.9) * (0.7) + (0.2) * (0.1) * (0.7)$$

$$= 0.2646 + 0.014$$

$$= 0.2786$$

FORWARD PROCEDURE (INDUCTION STEP)

Define $\alpha_t(i) = P(x_{1:t}, z_t = i)$

$$\alpha_{t+1}(j) = \left\{ \sum_i \alpha_t(i) * A_{ij} \right\} \phi_j(x_{t+1})$$

FORWARD PROCEDURE (INDUCTION STEP)

Define $\alpha_t(i) = P(x_{1:t}, z_t = i)$

$\alpha_1 = 0.42$ $\qquad$ $\alpha_2(B) = 0.2786$

$\boxed{Z_B}$ $\qquad$ $\boxed{Z_B}$

$\boxed{Z_F}$ $\qquad$ $\boxed{Z_F}$

$\alpha_1 = 0.2$ $\qquad$ $\alpha_2(F) = 0.1110$

FORWARD PROCEDURE (Termination)

$$\alpha_T(B) = P(x_{1:T}, Z_T = B)$$

$$\alpha_T(F) = P(x_{1:T}, Z_T = F)$$

$$\therefore \; P(x_{1:T}) = \alpha_T(B) + \alpha_T(F)$$

FORWARD PROCEDURE (Termination)

$$\alpha_T(B) = P(x_{1:T}, Z_T = B)$$

$$\alpha_T(F) = P(x_{1:T}, Z_T = F)$$

$$\therefore \; P(x_{1:T}) = \alpha_T(B) + \alpha_T(F)$$

In general,

$$P(x_{1:T} \mid \theta) = \sum_i \alpha_T(i)$$

or

$$L(X_{1:T} \mid \theta) = \sum_i \alpha_T(i)$$

P III HMM Filtering

Compute belief state

$p(z_t \mid x_{1:t})$ online (or recursively)

P III HMM Filtering

Compute belief state

$p(z_t \mid x_{1:t})$ online (or recursively)

$$= \frac{p(z_t, x_{1:t})}{p(x_{1:t})} = \frac{\alpha_t(i)}{\sum_i \alpha_t(i)}$$

P III HMM Filtering.

$$P(Z_1 = B \mid X_1 = H) = \frac{\alpha_1(B)}{\alpha_1(B) + \alpha_1(F)} = \frac{0.42}{0.62} \approx \frac{2}{3}$$

$$P(Z_1 = F \mid X_1 = H) = \frac{\alpha_1(F)}{\alpha_1(B) + \alpha_1(F)} = \frac{0.2}{0.62} \approx \frac{1}{3}$$

P III HMM Filtering.

$$P(Z_2 = B \mid X = \{HH\}) = \frac{\alpha_2(B)}{\alpha_2(B) + \alpha_2(F)} = 0.715$$

$$P(Z_2 = F \mid X = \{HH\}) = \frac{\alpha_2(F)}{\alpha_2(B) + \alpha_2(F)} = 0.289$$

P III HMM Filtering.

$$P(Z_2 = B \mid X = \{HH\}) = \frac{\alpha_2(B)}{\alpha_2(B) + \alpha_2(F)} = 0.715$$

$$P(Z_2 = F \mid X = \{HH\}) = \frac{\alpha_2(F)}{\alpha_2(B) + \alpha_2(F)} = 0.289$$

Note

$$P(Z_2 = B \mid \{H\,H\}) > P(Z_1 = B \mid \{H\})$$

$\because$ Biased coin $\Rightarrow$ more heads!

PIV : $P(X_{t+1:T} \mid Z_t = i, \theta)$?

Backwards algorithm

$$\beta_t(i) = P(X_{t+1:T} \mid Z_t = i, \theta)$$

= Probability of sequence $X_{t+1:T}$ given state is 'i' at t^{th} time instance

$\underline{P\overline{IV}}$: $P(X_{t+1:T} \mid Z_t = i, \theta)$?

$$\beta_t(i) = P(X_{t+1:T} \mid Z_t = i, \theta)$$

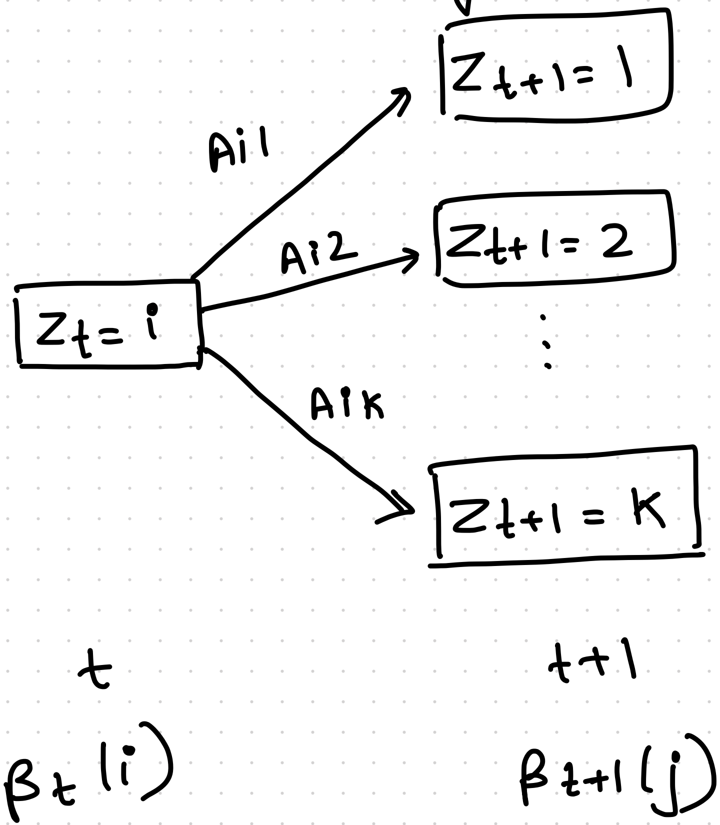
e.g. $X = \{H, H, H\}$

$T = 3$

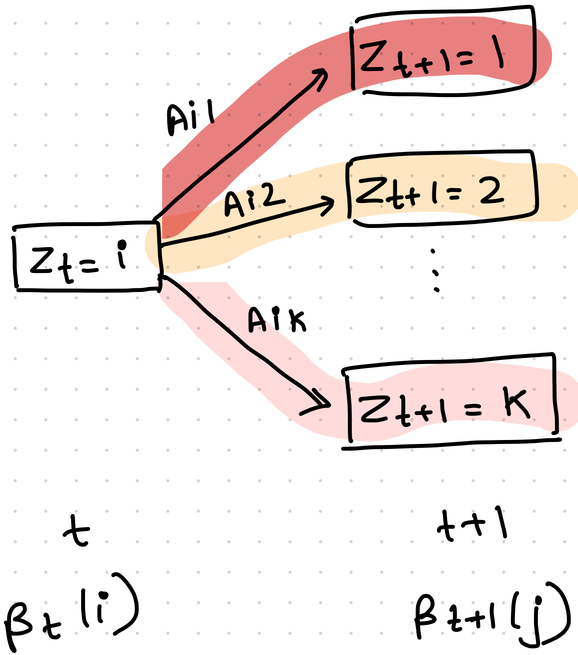
$t = 1$

$Z_1 =$ Biased

Backwards Algorithm



Backwards Algorithm



$$\beta_t(i) = P(x_{t+1:T} \mid z_t = i)$$

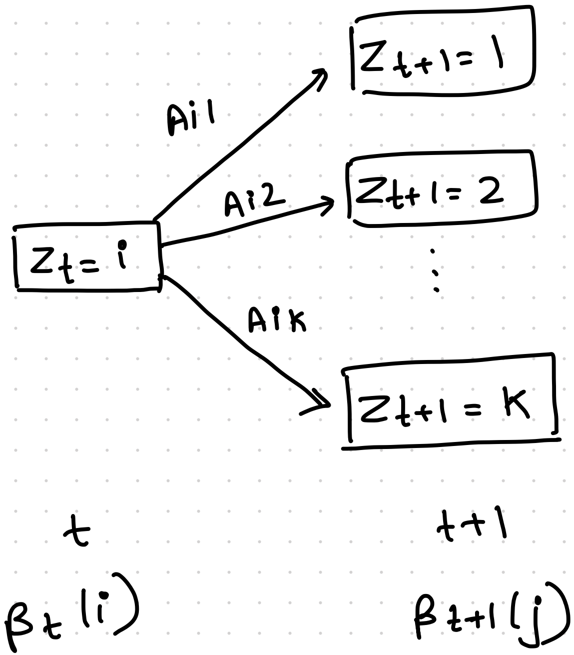
$$= P(x_{t+2:T} \mid z_{t+1} = 1) \cdot A_{i1} \cdot \phi_1(x_{t+1})$$

$$+ P(x_{t+2:T} \mid z_{t+1} = 2) \cdot A_{i2} \cdot \phi_2(x_{t+1})$$

$$\vdots$$

$$+ P(x_{t+2:T} \mid z_{t+1} = K) \cdot A_{iK} \cdot \phi_K(x_{t+1})$$

Backwards Algorithm

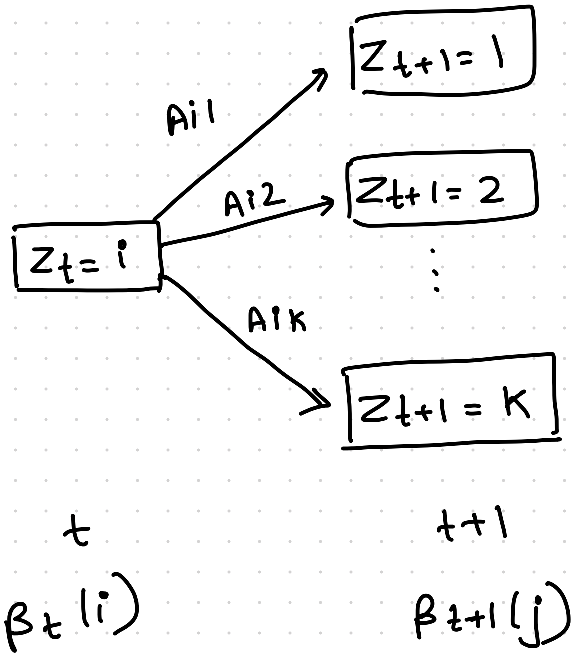


INDUCTION STEP

$$\beta_t(i) = P(x_{t+1:T} \mid z_t = i)$$

$$= \sum_{j=1}^{K} \beta_{t+1}(j) \cdot A_{ij} \cdot \phi_j(x_{t+1})$$

Backwards Algorithm



INDUCTION STEP

$$\beta_t(i) = P(x_{t+1:T} \mid z_t = i)$$

$$= \sum_{j=1}^{K} \beta_{t+1}(j) \cdot A_{ij} \cdot \phi_j(x_{t+1})$$

$t = T-1, \ldots 1$

INIT. STEP

$$\beta_T(i) = 1 \quad \forall i \in \{1, \ldots K\}$$

(Arbitrarily defined)

Backwards Algorithm (example)

$X = \{H, H, H\}$

Find $\beta_1, \beta_2, \beta_3$

INIT

$\beta_3(B) = 1 = \beta_3(F)$ (Arbitrary)

Backwards Algorithm (example)

$X = \{H, H, H\}$ Find $\beta_1, \beta_2, \beta_3$

INIT

$$\beta_3(B) = 1 = \beta_3(F) \quad \text{(Arbitrary)}$$

STEP

$$\beta_2(B) = \sum_{j=\{B,F\}} \beta_3(j) \cdot A_{Bj} \cdot \phi_j(H)$$

$$= \beta_3(B) . A_{BB} \cdot \phi_B(H) + \beta_3(F) . A_{BF} \cdot \phi_F(H)$$

$$= 1 \cdot (0 \cdot 9) \cdot (0 \cdot 7) + (1)(0 \cdot 1)(0 \cdot 5) = 0 \cdot 68$$

Backwards Algorithm (example)

$X = \{H, H, H\}$ Find $\beta_1, \beta_2, \beta_3$

INIT

$$\beta_3(B) = 1 = \beta_3(F) \quad \text{(Arbitrary)}$$

STEP

$$\beta_2(B) = \sum_{j=\{B,F\}} \beta_3(j) \cdot A_{Bj} \cdot \phi_j(H)$$

$$= \beta_3(B) \cdot A_{BB} \cdot \phi_B(H) + \beta_3(F) \cdot A_{BF} \cdot \phi_F(H)$$

$$= 1 \cdot (0.9) \cdot (0.7) + (1)(0.1)(0.5) = 0.68$$

$$\beta_2(F) = \beta_3(B) \cdot A_{FB}\, \phi_B(H) + \beta_3(F) \cdot A_{FF} \cdot \phi_F(H)$$

$$= (1) \cdot (0.1)(0.7) + (1)(0.9)(0.5) = 0.52$$

Backwards Algorithm (example)

$X = \{H, H, H\}$ Find $\beta_1, \beta_2, \beta_3$

INIT

$\beta_3(B) = 1 = \beta_3(F)$ (Arbitrary)

STEP

$$\beta_1(B) = \beta_2(B) . A_{BB} . \phi_B(H) + \beta_2(F) . A_{BF} . \phi_F(H)$$
$$= (0.68)(0.9)(0.7) + (0.52).(0.1)(0.5) = 0.4544$$

$$\beta_1(F) = \beta_2(B) . A_{FB} . \phi_B(H) + \beta_2(F) . A_{FF} . \phi_F(H)$$
$$= (0.68)(0.1)(0.7) + (0.52)(0.9)(0.5) = 0.2816$$

$\underline{PV}$ "OPTIMAL" sequence of states given model and observations.

Given $X = \{X_1, \dots X_T\}$

$$\Theta = \{\pi, A, \phi\}$$

How to choose $Z = \{Z_1, \dots Z_T\}$ optimally?

<u>PV</u> 'OPTIMAL' sequence of states : MPM

OPTIMALITY DEFINITION #1

Choose state z_t which is individually most likely

PV "OPTIMAL" sequence of states : MPM

OPTIMALITY DEFINITION #1

Choose state z_t which is individually most likely

$$\hat{z} = \left(\underset{z_1}{\arg\max}\ p(z_1 \mid x_{1:T}), \ldots, \underset{z_T}{\arg\max}\ p(z_T \mid x_{1:T}) \right)$$

PV "OPTIMAL" sequence of states : MPM

OPTIMALITY DEFINITION #1

Choose state z_t which is individually most likely

$$\hat{z} = \left(\underset{z_1}{\arg\max}\, p(z_1 \mid x_{1:T}), \ldots, \underset{z_T}{\arg\max}\, p(z_T \mid x_{1:T}) \right)$$

Also called Sequence of Marginally Most Probable States (MPM)

PV 'OPTIMAL' sequence of states : MPM

OPTIMALITY DEFINITION #1

Choose state z_t which is individually most likely

Let $\gamma_t(i) = p(z_t = i \mid x_{1:T})$

PV "OPTIMAL" sequence of states : MPM

OPTIMALITY DEFINITION #1

Choose state z_t which is individually most likely

Let $\gamma_t(i) = p(z_t = i \mid x_{1:T})$

$$\propto p(z_t = i \mid x_{1:t}) \cdot p(x_{t+1:T} \mid z_t = i, x_{1:t})$$

<u>PV</u> 'OPTIMAL' sequence of states : MPM

OPTIMALITY DEFINITION #1

Choose state z_t which is individually most likely

Let $\gamma_t(i) = p(z_t = i \mid x_{1:T})$

$$\propto p(z_t = i \mid x_{1:t}) \cdot p(x_{t+1:T} \mid z_t = i, \cancel{x_{1:t}})$$

Independent

$$\propto p(z_t = i \mid x_{1:t}) \cdot p(x_{t+1:T} \mid z_t = i)$$

$$\gamma_t(i) \propto \alpha_t(i) \cdot \beta_t(i)$$

<u>PV</u> "OPTIMAL" sequence of states : MPM

FORWARD BACKWARD ALGORITHM

$$\gamma_t(i) = p(z_t = i \mid x_{1:T})$$

$$\gamma_t(i) \propto \alpha_t(i) . \beta_t(i)$$

$$\gamma_t(i) = \frac{\alpha_t(i) . \beta_t(i)}{\sum_i \alpha_t(i) . \beta_t(i)}$$

$\overline{PV}$ 'OPTIMAL' sequence of states : MPM

FORWARD BACKWARD ALGORITHM

Example: For some Θ, $X = \{H, H, H\}$ Find 'smooth' set of states

$$\gamma_t(i) = \frac{\alpha_t(i) \cdot \beta_t(i)}{\sum_i \alpha_t(i) \cdot \beta_t(i)}$$

$\overline{\text{PV}}$ "OPTIMAL" sequence of states : MPM

FORWARD BACKWARD ALGORITHM

Example: For some Θ, $X = \{H, H, H\}$ Find 'smooth' set of states

$$\gamma_t(i) = \frac{\alpha_t(i) \cdot \beta_t(i)}{\sum_i \alpha_t(i) \cdot \beta_t(i)}$$

$$\gamma_1(B) = \frac{\alpha_1(B) \cdot \beta_1(B)}{\alpha_1(B) \cdot \beta_1(B) + \alpha_1(F) \cdot \beta_1(F)} = \frac{0.42 \times 0.4544}{.42 \times 0.4544 + 0.2 \times 0.2816}$$

$$= \frac{.190848}{.247168} = 0.77$$

PV "OPTIMAL" sequence of states : MPM

FORWARD BACKWARD ALGORITHM

Example: For some θ, $X = \{H, H, H\}$ Find 'smooth' set of states

$$\gamma_t(i) = \frac{\alpha_t(i) \cdot \beta_t(i)}{\sum_i \alpha_t(i) \cdot \beta_t(i)}$$

$\gamma_1(B) = 0.77$

$\Rightarrow \gamma_1(F) = 0.23$

Much more likely to have $z_1 = B$ if $\{H, H, H\}$ and θ given

$\overline{PV}$ 'OPTIMAL' sequence of states : MPM

FORWARD BACKWARD ALGORITHM

Example: For some Θ, $X = \{H, H, H\}$ Find 'smooth' set of states

$$\gamma_t(i) = \frac{\alpha_t(i) \cdot \beta_t(i)}{\sum_i \alpha_t(i) \cdot \beta_t(i)}$$

$$\hat{Z} = \left(\underset{i}{\operatorname{argmax}}\ \gamma_1(i), \ldots\right) = (\text{Biased}, \text{Biased}, \text{Biased})$$

$\overline{\text{PV}}$ 'OPTIMAL' sequence of states : MAP

OPTIMALITY DEFINITION # 2

CHOOSE MOST PROBABLE SEQUENCE OF STATES

$$z^* = \underset{z_{1:T}}{\arg\max}\; p(z_{1:T} \mid x_{1:T})$$

$\overline{PV}$ 'OPTIMAL' sequence of states : MAP v/s MPM

CONSIDER FOLLOWING 2 timesteps joint prob

	$Z_1 = 1$	$Z_1 = 2$
$Z_2 = 1$	0.04	0.3
$Z_2 = 2$	0.36	0.3

$\overline{PV}$ "OPTIMAL" sequence of states : MAP v/s MPM

CONSIDER FOLLOWING 2 timesteps joint prob

	$Z_1 = 1$	$Z_1 = 2$	
$Z_2 = 1$	0.04	0.3	0.34
$Z_2 = 2$	0.36	0.3	0.66
	.4	.6	

MPM = sequence of Marginally MOST PROB. STATES

$$= \left(Z_1 = 2 \quad , \quad Z_2 = 2 \right)$$

$\because 0.6 > 0.4$ $\qquad \because 0.66 > 0.34$

PV "OPTIMAL" sequence of states : MAP v/s MPM

CONSIDER FOLLOWING 2 timesteps joint prob

	$z_1 = 1$	$z_1 = 2$	
$z_2 = 1$	0.04	0.3	0.34
$z_2 = 2$	0.36	0.3	0.66
	.4	.6	

MAP = $(z_1 = 1, z_2 = 2)$

$\because$ .36 is highest number

<u>PV</u> 'OPTIMAL' sequence of states : MAP

OPTIMALITY DEFINITION # 2

CHOOSE MOST PROBABLE SEQUENCE OF STATES

$$z^* = \underset{z_{1:T}}{\arg\max}\; p(z_{1:T} \mid x_{1:T})$$

VITERBI ALGORITHM (Dynamic Programming)

VITERBI ALGORITHM

Define $\delta_t(i) = \max\limits_{z_1, z_2, \ldots z_{t-1}} P\left[z_1, z_2, z_3 \ldots, z_{t-1}, z_t = i, x_1, x_2 \ldots x_t \mid \theta\right]$

VITERBI ALGORITHM

Define $\delta_t(i) = \max_{z_1, z_2, \ldots z_{t-1}} P[z_1, z_2, z_3 \ldots, z_{t-1}, z_t = i, x_1, x_2 \ldots x_t \mid \theta]$

Best score (highest prob.) along a single path at time t, which accounts for first t observations and ends in $z_t = i$

Difference B/w $\alpha_t(i)$ and $\delta_t(i)$

$$\delta_t(i) = \max_{z_1, z_2, \dots z_{t-1}} P\left[z_1, z_2, z_3 \dots, z_{t-1}, z_t = i, x_1, x_2 \dots x_t \mid \theta\right]$$

$$\alpha_t(i) = P\left(x_{1:t}, z_t = i \mid \theta\right)$$

Difference B/w $\alpha_t(i)$ and $\delta_t(i)$

$$\delta_t(i) = \max_{z_1, z_2, \ldots z_{t-1}} P\left[z_1, z_2, z_3 \ldots, z_{t-1}, z_t = i, x_1, x_2 \ldots x_t \mid \theta\right]$$

$$\alpha_t(i) = P\left(x_{1:t}, z_t = i \mid \theta\right)$$

FOCUS ON MOST LIKELY SEQUENCE

FOCUS ON MOST LIKELY STATE AT 't'

VITERBI ALGORITHM

Define $\delta_t(i) = \max\limits_{z_1, z_2, \ldots z_{t-1}} P\left[z_1, z_2, z_3 \ldots, z_{t-1}, z_t = i, x_1, x_2 \ldots x_t \mid \theta\right]$

Relation b/w $\delta_t(i)$ AND $\delta_{t+1}(j)$?

VITERBI ALGORITHM

Define $$\delta_t(i) = \max_{z_1, z_2, \dots z_{t-1}} P\left[z_1, z_2, z_3 \dots, z_{t-1}, z_t = i, x_1, x_2 \dots x_t \,\middle|\, \theta\right]$$

Relation b/w $\delta_t(i)$ AND $\delta_{t+1}(j)$?

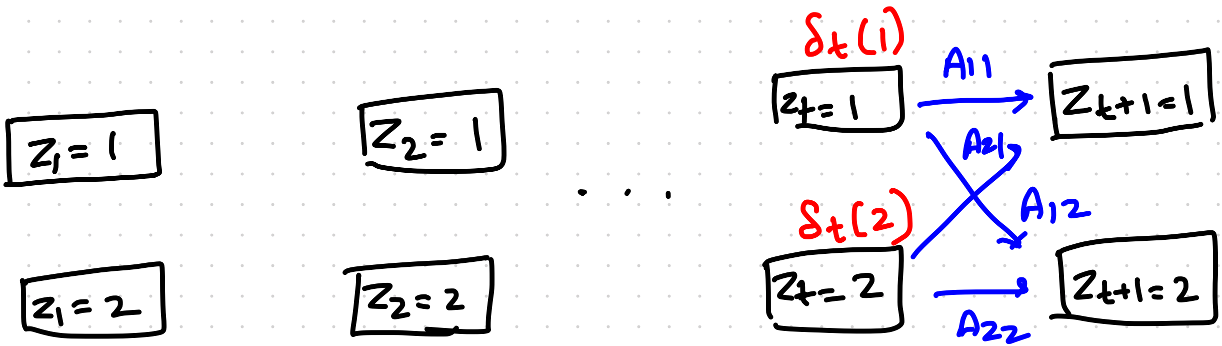
$$\boxed{z_1 = 1} \quad \boxed{z_2 = 1} \quad \dots \quad \boxed{z_t = 1} \quad \boxed{z_{t+1} = 1}$$

$$\boxed{z_1 = 2} \quad \boxed{z_2 = 2} \quad \quad \boxed{z_t = 2} \quad \boxed{z_{t+1} = 2}$$

VITERBI ALGORITHM

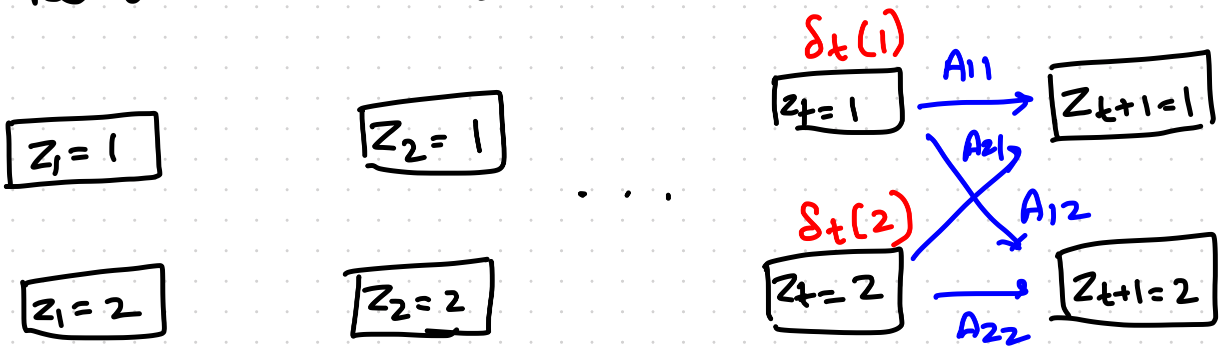
Define $$\delta_t(i) = \max_{z_1, z_2, \dots z_{t-1}} P\left[z_1, z_2, z_3 \dots, z_{t-1}, z_t = i, x_1, x_2 \dots x_t \mid \theta\right]$$

Relation b/w $\delta_t(i)$ AND $\delta_{t+1}(j)$?



VITERBI ALGORITHM

Relation b/w $\delta_t(i)$ AND $\delta_{t+1}(j)$?

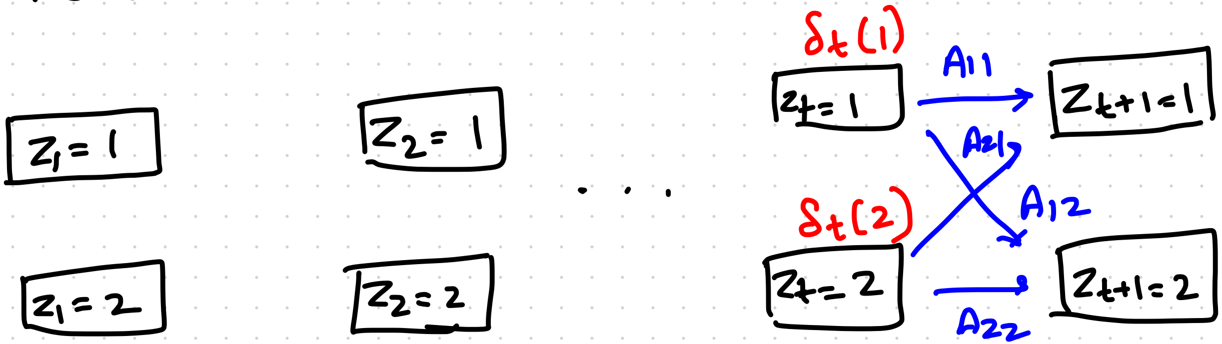


* We could reach $z_{t+1} = j$ from any $i \in \{1, .. K\}$ via a transition with prob. A_{ij}

* Once we reach $z_{t+1} = j$, prob. of observing x_{t+1} is $\phi_j(x_{t+1})$

VITERBI ALGORITHM

Relation b/w $\delta_t(i)$ AND $\delta_{t+1}(j)$?



* We could reach $z_{t+1} = j$ from any $i \in \{1, .. K\}$ via a transition with prob. A_{ij}. We take 'best path'
* Once we reach $z_{t+1} = j$, prob. of observing x_{t+1} is $\phi_j(x_{t+1})$

$$\delta_{t+1}(j) = \left[\max_i \{ \delta_t(i) \cdot A_{ij} \} * \phi_j(x_{t+1}) \right]$$

VITERBI ALGORITHM

$$\delta_{t+1}(j) = \left[\max_i \{\delta_t(i) \cdot A_{ij}\} * \phi_j(x_{t+1})\right]$$

FOR EACH t and j, we need to store argument 'i' which maximized above equation.

called $\psi_t(j)$

VITERBI ALGORITHM

(i) INITIALISATION

$$\delta_1(i) =$$

VITERBI ALGORITHM

(i) INITIALISATION

$$\delta_1(i) = \pi_i * \phi_i(x_1)$$

$$\psi_1(i) = 0 \quad \text{(ARBITRARY)}$$

VITERBI ALGORITHM

① INITIALISATION

$$\delta_1(i) = \pi_i * \phi_i(x_1)$$

$$\psi_1(i) = 0 \quad \text{(ARBITRARY)}$$

② RECURSION

FOR t in 2 ~~to~~ T

FOR j in 1 ~~to~~ K

$$\delta_t(j) = \left(\max_{i \in \{1,K\}} \left(\delta_{t-1}(i) \cdot A_{ij}\right)\right) \phi_j(x_t)$$

$$\psi_t(j) = \underset{i \in \{1,..K\}}{\operatorname{argmax}} \; \delta_{t-1}(i) \cdot A_{ij}$$

VITERBI ALGORITHM

③ Termination

$$P^* = \max_{i \in \{1,..K\}} \delta_T(i)$$

$$Z_T^* = \underset{i \in \{1,..K\}}{\operatorname{argmax}} \delta_T(i)$$

VITERBI ALGORITHM

③ Termination

$$P^* = \max_{i \in \{1,..K\}} \delta_T(i)$$

$$Z_T^* = \underset{i \in \{1,..,K\}}{\operatorname{argmax}} \delta_T(i)$$

④ Backtracking

For t in T-1, ... 1 :

$$Z_t^* = \Psi_{t+1}(Z_{t+1}^*)$$

VITERBI ALGORITHM

Example

Given $\Theta = \{\pi, A, \phi\}$ and $x_{1:T} = \{H, H, H\}$

determine MAP z^*

$$\pi = \begin{array}{c} \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \\ \text{Bias} \quad \text{Fair} \end{array}$$

$$A = \begin{array}{c|cc} & B & F \\ \hline \text{Bias} & 0.9 & 0.1 \\ \text{Fair} & 0.1 & 0.9 \end{array}$$

$\phi =$

Bias	Fair
$p(H) = 0.7$	$p(H) = 0.5$
$p(T) = 0.3$	$p(T) = 0.5$

VITERBI ALGORITHM

Example

Given $\Theta = \{\pi, A, \phi\}$ and $x_{1:T} = \{H, H, H\}$

determine MAP z^*

$$\delta_1(\text{Bias}) = \pi_{\text{Bias}} * \phi_{\text{Bias}}(H) = 0.6 * 0.7 = 0.42$$

$$\delta_1(\text{Fair}) = \pi_{\text{Fair}} * \phi_{\text{Fair}}(H) = 0.4 * 0.5 = 0.20$$

VITERBI ALGORITHM

Example

Given $\Theta = \{\pi, A, \phi\}$ and $x_{1:T} = \{H, H, H\}$

determine MAP z^*

$$\delta_1(\text{Bias}) = \pi_{\text{Bias}} * \phi_{\text{Bias}}(H) = 0.6 * 0.7 = 0.42$$

$$\delta_1(\text{Fair}) = \pi_{\text{Fair}} * \phi_{\text{Fair}}(H) = 0.4 \times 0.5 = 0.20$$

$$\psi_1(\text{Bias}) = \psi_1(\text{Fair}) = 0 \text{ (Arbitrary)}$$

VITERBI ALGORITHM

Example

$$\delta_1(\text{Bias}) = \pi_{\text{Bias}} * \phi_{\text{Bias}}(H) = 0.6 * 0.7 = 0.42$$

$$\delta_1(\text{Fair}) = \pi_{\text{Fair}} * \phi_{\text{Fair}}(H) = 0.4 * 0.5 = 0.20$$

$$\psi_1(\text{Bias}) = \psi_1(\text{Fair}) = 0 \quad (\text{Arbitrary})$$

$$\delta_2(\text{Bias}) = \max \begin{Bmatrix} \delta_1(\text{Fair}) . A_{\text{Fair, Bias}} \\ \delta_1(\text{Bias}) . A_{\text{Bias, Bias}} \end{Bmatrix} * \phi_{\text{Bias}}(H)$$

$$= \max \begin{Bmatrix} 0.2 * 0.1 \\ 0.42 * 0.9 \end{Bmatrix} * 0.7 = 0.42 * 0.9 * 0.7$$

VITERBI ALGORITHM

$$\delta_1(\text{Bias}) = \pi_{\text{Bias}} * \phi_{\text{Bias}}(H) = 0.6 * 0.7 = 0.42$$

$$\delta_1(\text{Fair}) = \pi_{\text{Fair}} * \phi_{\text{Fair}}(H) = 0.4 * 0.5 = 0.20$$

$$\psi_1(\text{Bias}) = \psi_1(\text{Fair}) = 0 \text{ (Arbitrary)}$$

$$\delta_2(\text{Bias}) = \max \left\{ \begin{array}{l} \delta_1(\text{Fair}) . A_{\text{Fair, Bias}} \\ \delta_1(\text{Bias}) . A_{\text{Bias, Bias}} \end{array} \right\} * \phi_{\text{Bias}}(H)$$

$$= \max \left\{ \begin{array}{l} 0.2 * 0.1 \\ 0.42 * 0.9 \end{array} \right\} * 0.7 = 0.42 * 0.9 * 0.7$$

$$= 0.26$$

$$\psi_2(\text{Bias}) = \text{Bias}$$

VITERBI ALGORITHM

$$\delta_1(\text{Bias}) = \pi_{\text{Bias}} * \phi_{\text{Bias}}(H) = 0.6 * 0.7 = 0.42$$

$$\delta_1(\text{Fair}) = \pi_{\text{Fair}} * \phi_{\text{Fair}}(H) = 0.4 * 0.5 = 0.20$$

$$\psi_1(\text{Bias}) = \psi_1(\text{Fair}) = 0 \quad (\text{Arbitrary})$$

$$\delta_2(\text{Fair}) = \max \begin{Bmatrix} \delta_1(\text{Fair}) . A_{\text{Fair},\text{Fair}} \\ \delta_1(\text{Bias}) . A_{\text{Bias},\text{Fair}} \end{Bmatrix} * \phi_{\text{Fair}}(H)$$

$$= \max \begin{Bmatrix} 0.2 * 0.9 \\ 0.42 * 0.1 \end{Bmatrix} * 0.5 = 0.2 * 0.9 * 0.5$$

$$= 0.09$$

$$\psi_2(\text{Fair}) = \text{Fair}$$

VITERBI ALGORITHM

$\delta_2(\text{Bias}) = 0.2646$ $\qquad$ $\psi_2(\text{Bias}) = \text{Bias}$

$\delta_2(\text{Fair}) = 0.09$ $\qquad$ $\psi_2(\text{Fair}) = \text{Fair}$

$$\delta_3(\text{Bias}) = \max \begin{Bmatrix} \delta_2(\text{Bias}) * A_{\text{Bias},\text{Bias}} \\ \delta_2(\text{Fair}) * A_{\text{Fair},\text{Bias}} \end{Bmatrix} * \phi_{\text{Bias}}(H)$$

$$\delta_3(\text{Bias}) = \max \begin{Bmatrix} .2646 * 0.9 \\ 0.09 * 0.1 \end{Bmatrix} * 0.7 = 0.166698$$

$\psi_3(\text{Bias}) = \text{Bias}$

VITERBI ALGORITHM

$\delta_2(\text{Bias}) = 0.2646$ $\quad\quad$ $\psi_2(\text{Bias}) = \text{Bias}$

$\delta_2(\text{Fair}) = 0.09$ $\quad\quad$ $\psi_2(\text{Fair}) = \text{Fair}$

$$\delta_3(\text{Fair}) = \max \begin{Bmatrix} \delta_2(\text{Fair}) \cdot A_{\text{Fair},\text{Fair}} \\ \delta_2(\text{Bias}) \cdot A_{\text{Bias},\text{Fair}} \end{Bmatrix} * \phi_{\text{Fair}}(H)$$

$$= \max \begin{Bmatrix} 0.09 * 0.9 \\ 0.2646 * 0.1 \end{Bmatrix} * 0.5 = \max \begin{Bmatrix} 0.081 \\ 0.026 \end{Bmatrix} * 0.5$$

$$= 0.0405$$

$\psi_3(\text{Fair}) = \text{Fair}$

VITERBI ALGORITHM

$\delta_1(\text{Bias}) = 0.42$

$\delta_1(\text{Fair}) = 0.2$

$\delta_2(\text{Bias}) = 0.2646$

$\delta_2(\text{Fair}) = 0.09$

$\psi_2(\text{Bias}) = \text{Bias}$

$\psi_2(\text{Fair}) = \text{Fair}$

$\delta_3(\text{Bias}) = 0.167$

$\delta_3(\text{Fair}) = 0.04$

$\psi_3(\text{Bias}) = \text{Bias}$

$\psi_3(\text{Fair}) = \text{Fair}$

VITERBI ALGORITHM

$\delta_1(\text{Bias}) = 0.42$

$\delta_1(\text{Fair}) = 0.2$

$\delta_2(\text{Bias}) = 0.2646$ $\quad\quad \psi_2(\text{Bias}) = \text{Bias}$

$\delta_2(\text{Fair}) = 0.09$ $\quad\quad \psi_2(\text{Fair}) = \text{Fair}$

$\delta_3(\text{Bias}) = 0.167$ $\quad\quad \psi_3(\text{Bias}) = \text{Bias}$

$\delta_3(\text{Fair}) = 0.04$ $\quad\quad \psi_3(\text{Fair}) = \text{Fair}$

$$Z_3^* = \operatorname{argmax}\left[\delta_3(\text{Bias}), \delta_3(\text{Fair})\right] = \text{Bias}$$

VITERBI ALGORITHM

$\delta_1(\text{Bias}) = 0.42$

$\delta_1(\text{Fair}) = 0.2$

$\delta_2(\text{Bias}) = 0.2646$

$\delta_2(\text{Fair}) = 0.09$

$\delta_3(\text{Bias}) = 0.167$

$\delta_3(\text{Fair}) = 0.04$

$\psi_2(\text{Bias}) = \text{Bias}$

$\psi_2(\text{Fair}) = \text{Fair}$

$\psi_3(\text{Bias}) = \text{Bias}$

$\psi_3(\text{Fair}) = \text{Fair}$

$$z_3^* = \operatorname{argmax}\left[\delta_3(\text{Bias}), \delta_3(\text{Fair})\right] = \text{Bias}$$

$$z_2^* = \psi_3(\text{Bias}) = \text{Bias}$$

$$z_1^* = \psi_2(z_2^*) = \psi_2(\text{Bias}) = \text{Bias} \Rightarrow z^* = \{\text{Bias}, \text{Bias}, \text{Bias}\}$$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

* Fully observed: Observe all hidden state sequences

* $\theta = \{\pi, A, \phi\}$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

* Fully observed: observe all hidden state sequences

* $\theta = \{\pi, A, \phi\}$

* Given $D = \Big\{ \big((z_{11}, z_{12} \ldots z_{1T_1}), (x_{11}, x_{12} \ldots x_{1T_1})\big),$

$\vdots$

$\big((z_{N1}, \ldots z_{NT_N}), (x_{N1}, \ldots x_{NT_N})\big)$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

* Fully observed: observe all hidden state sequences

* $\theta = \{\pi, A, \phi\}$

* Given $D = \Big\{ \big((z_{11}, z_{12} \dots z_{1T_1}), (x_{11}, x_{12} \dots x_{1T_1})\big),$

$\vdots$

N sequences $\longleftarrow$

$\big((z_{N1}, \dots z_{NT_N}), (x_{N1}, \dots x_{NT_N})\big)$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

* Fully observed: observe all hidden state sequences

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* Given $D = \Big\{ \big((z_{11}, z_{12} \dots z_{1T_1}), (x_{11}, x_{12} \dots x_{1T_1})\big),$

$\vdots$

N sequences each of T_i length (may be different) ←

$\big((z_{N1}, \dots z_{NT_N}), (x_{N1}, \dots x_{NT_N})\big)$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

SOME MATHS FOR A SINGLE SEQUENCE FIRST

Likelihood of a sequence $Z_{1:T}$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

Likelihood of a sequence $z_{1:T}$

$$= \pi(z_1)\, A(z_1, z_2) \cdots A(z_{T-1}, z_T)$$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

Likelihood of a sequence $z_{1:T}$

$$= \pi(z_1)\, A(z_1, z_2) \;\cdots\; A(z_{T-1}, z_T)$$

Let us write $\pi(z_1)$ in terms of $j \in \{1, .. K\}$

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Likelihood of a sequence $z_{1:T}$

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Let us write $\pi(z_1)$ in terms of $j \in \{1, .. K\}$

$$\pi = \underset{z_1=1}{[\text{----}}\;\underset{z_1=2}{\text{-----}}\;\underset{z_1=j}{\boxed{}}\;\underset{z_1=K}{\text{----}]}$$

$$\pi(z_1) = \pi_j \quad : \text{if } z_1 = j$$

$\underline{\text{PVI}}$: HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

Likelihood of a sequence $z_{1:T}$

$$= \pi(z_1)\, A(z_1, z_2) \;\ldots\; A(z_{T-1}, z_T)$$

Let us write $\pi(z_1)$ in terms of $j \in \{1, .. K\}$

$$\pi = \underset{z_1 = 1 \quad z_1 = 2 \quad z_1 = j \quad z_1 = K}{[\text{-----------------------}]}$$

$$\pi(z_1) = \pi_j \;:\; \text{if } z_1 = j \qquad \text{or} \qquad \pi(z_1) = \prod_{j=1}^{K} (\pi_j)^{I(z_1 = j)}$$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

Likelihood of a sequence $z_{1:T}$

$$= \pi(z_1)\, A(z_1, z_2) \cdots A(z_{T-1}, z_T)$$

Let us write $A(z_1, z_2)$ in terms of $j, k \in \{1, \dots K\}$

$\underline{\text{PVI}}$: HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

Likelihood of a sequence $z_{1:T}$

$$= \pi(z_1) A(z_1, z_2) \ldots A(z_{T-1}, z_T)$$

Let us write $A(z_1, z_2)$ in terms of $j, k \in \{1, \ldots K\}$

$$A(z_1, z_2) = \prod_{j=1}^{K} \prod_{k=1}^{K} (A_{jk})^{I(z_1 = j, z_2 = k)}$$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

Likelihood of a sequence $z_{1:T}$

$$= \pi(z_1)\, A(z_1, z_2) \cdots A(z_{T-1}, z_T)$$

Now, $A(z_1, z_2) = \prod_{j=1}^{K} \prod_{k=1}^{K} (A_{jk})^{\mathbb{I}(z_1 = j,\, z_2 = k)}$

$$\Rightarrow A(z_1, z_2) \cdots A(z_{T-1}, z_T) = \prod_{t=2}^{T} \prod_{j=1}^{K} \prod_{k=1}^{K} (A_{jk})^{\mathbb{I}(z_t = k,\, z_{t-1} = j)}$$

$\underline{\text{PVI}}$: HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

Likelihood of a sequence $z_{1:T}$

$$= \pi(z_1)\, A(z_1, z_2) \ldots A(z_{T-1}, z_T)$$

$$LL(D \mid \theta = \{A, \pi\}) = \log \left(\prod_{j=1}^{K} (\pi_j)^{I(z_1 = j)} \prod_{t=2}^{T} \prod_{j=1}^{K} \prod_{k=1}^{K} (A_{jk})^{I(z_t = k,\, z_{t-1} = j)} \right)$$

$$= \log (F \cdot G)$$

$$= \log F + \log G$$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

$$LL(D \mid \theta = \{A, \pi\}) = \log\left(\prod_{j=1}^{K} (\pi_j)^{I(z_1 = j)} \prod_{t=2}^{T} \prod_{j=1}^{K} \prod_{k=1}^{K} (A_{jk})^{I(z_t = k,\, z_{t-1} = j)}\right)$$

$$= \log(F \cdot G)$$

$$= \log F + \log G$$

Now $\log F = \log\left(\pi_1^{I(z_1=1)}\, \pi_2^{I(z_1=2)} \cdots \pi_K^{I(z_1=K)}\right)$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

$$LL(D \mid \theta = \{A, \pi\}) = \log \left(\prod_{j=1}^{K} (\pi_j)^{I(z_1 = j)} \prod_{t=2}^{T} \prod_{j=1}^{K} \prod_{k=1}^{K} (A_{jk})^{I(z_t = k,\, z_{t-1} = j)} \right)$$

$$= \log (F \cdot G)$$

$$= \log F + \log G$$

Now $\log F = \log \left(\pi_1^{I(z_1 = 1)} \pi_2^{I(z_1 = 2)} \cdots \pi_K^{I(z_1 = K)} \right)$

$$= I(z_1 = 1) \log \pi_1 + \cdots$$

$$\log F = \sum_{j=1}^{K} I(z_1 = j) \log \pi_j$$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

$$LL(D \mid \theta = \{A, \pi\}) = \log \left(\prod_{j=1}^{K} (\pi_j)^{I(z_1 = j)} \prod_{t=2}^{T} \prod_{j=1}^{K} \prod_{k=1}^{K} (A_{jk})^{I(z_t = k,\, z_{t-1} = j)} \right)$$

$$= \log (F \cdot G)$$

$$\log G = \log \left(\prod_{j=1}^{K} \prod_{k=1}^{K} A_{jk}^{I(z_1 = j,\, z_2 = k)} * \prod_{j=1}^{K} \prod_{k=1}^{K} A_{jk}^{I(z_2 = j,\, z_3 = k)} \cdots \right)$$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

$$LL(D \mid \theta = \{A, \pi\}) = \log \left(\prod_{j=1}^{K} (\pi_j)^{I(z_1 = j)} \prod_{t=2}^{T} \prod_{j=1}^{K} \prod_{k=1}^{K} (A_{jk})^{I(x_t = k,\, x_{t-1} = j)} \right)$$

$$= \log (F \cdot G)$$

$$\log G = \log \left(\prod_{j=1}^{K} \prod_{k=1}^{K} A_{jk}^{I(x_1 = j,\, x_2 = k)} * \prod_{j=1}^{K} \prod_{k=1}^{K} A_{jk}^{I(x_2 = j,\, x_3 = k)} \; \text{----} \right)$$

$$= \sum_{t=2}^{T} \log \left(\prod_{j=1}^{K} \prod_{k=1}^{K} A_{jk}^{I(x_t = k,\, x_{t-1} = j)} \right)$$

$\underline{\text{PVI}}$: HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

$$\log G = \log\left(\prod_{j=1}^{K}\prod_{k=1}^{K} A_{jk}^{I(x_1=j,\,x_2=k)} * \prod_{j=1}^{K}\prod_{k=1}^{K} A_{jk}^{I(x_2=j,\,x_3=k)} \cdots\right)$$

$$= \sum_{t=2}^{T} \log\left(\prod_{j=1}^{K}\prod_{k=1}^{K} A_{jk}^{I(x_t=k,\,x_{t-1}=j)}\right)$$

$$\log G = \sum_{t=2}^{T}\sum_{j=1}^{K}\sum_{k=1}^{K} I(x_t=k,\, x_{t-1}=j)\log A_{jk}$$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

BACK TO 'N' sequences now.

* Given $D = \{ ((z_{11}, z_{12} \ldots z_{1T_1}), (x_{11}, x_{12} \ldots x_{1T_1})),$

$\vdots$

$((z_{N1}, \ldots z_{NT_N}), (x_{N1}, \ldots x_{NT_N}))$

N sequences each of T_i length (may be different) ←

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

STEP I

Estimate π and A given $D' = ((z_{i1} \ldots z_{iT_i}) \; \forall i \in (1, N))$

$$\log p(D' \mid \theta' = \{A, \pi\}) = \sum_{i=1}^{N} \log p(z_i \mid \theta')$$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

STEP I

Estimate π and A given $D' = ((z_{i1} \ldots z_{iT_i})\ \forall\, i \in (1, N))$

$$\log p(D' \mid \theta' = \{A, \pi\}) = \sum_{i=1}^{N} \log p(z_i \mid \theta')$$

$$\boxed{\begin{aligned} N'_j &\triangleq \sum_{i=1}^{N} \mathbb{I}(z_{i1} = j) \\ N_{jk} &\triangleq \sum_{i=1}^{N} \sum_{t=1}^{T_i - 1} \mathbb{I}(x_{i,t} = j,\ x_{i,t+1} = k) \end{aligned}}$$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

STEP I

Estimate π and A given $D' = ((z_{i1} \ldots\ldots z_{iT_i})\ \forall\, i \in (1, N))$

$$\log p(D' \mid \theta' = \{A, \pi\}) = \sum_{i=1}^{N} \log p(z_i \mid \theta')$$

$$N_j^{1} \triangleq \sum_{i=1}^{N} \mathbb{I}(z_{i1} = j)$$ ← COUNT OF SEQUENCES WHERE $z_1 = j$

$$N_{jk} \triangleq \sum_{i=1}^{N} \sum_{t=1}^{T_i - 1} \mathbb{I}(x_{i,t} = j,\ x_{i,t+1} = k)$$

PVI: HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

STEP I

Estimate π and A given $D' = ((z_{i1} \ldots z_{iT_i}) \; \forall i \in (1, N))$

$$\log p(D' \mid \theta' = \{A, \pi\}) = \sum_{i=1}^{N} \log p(z_i \mid \theta')$$

$$N_j^1 \triangleq \sum_i I(z_{i1} = j)$$ ← COUNT OF SEQUENCES WHERE $z_1 = j$

COUNT OF TRANSITIONS FROM 'j' to 'k' →

$$N_{jk} \triangleq \sum_{i=1}^{N} \sum_{t=1}^{T_i - 1} I(x_{i,t} = j, \; x_{i,t+1} = k)$$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

FROM MLE,

$$\hat{\pi}_j = \frac{N^1_j}{\sum_{j=1}^{K} N^1_j} \qquad \hat{A}_{jk} = \frac{N_{jk}}{\sum_{k} N_{jk}}$$

$$N^1_j \triangleq \mathbb{I}(z_{i1} = j)$$

← COUNT OF SEQUENCES WHERE $z_1 = j$

COUNT OF TRANSITIONS FROM 'j' to 'k' →

$$N_{jk} \triangleq \sum_{i=1}^{N} \sum_{t=1}^{T_i-1} \mathbb{I}(x_{i,t} = j,\ x_{i,t+1} = k)$$

PVI : HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

FROM MLE,

$$\hat{\pi}_j = \frac{N_j^1}{\sum_{j=1}^{K} N_j^1} \qquad \hat{A}_{jk} = \frac{N_{jk}}{\sum_k N_{jk}}$$

HOW TO ESTIMATE ϕ:

① Assume MULTINOULLI EMISSION

$$\Rightarrow \phi_{jl} = p(x_t = l \mid z_t = j) \quad l \in \{1 \ldots L\}$$

$\overline{\text{PVI}}$: HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

FROM MLE,

$$\hat{\pi}_j = \frac{N_j^1}{\sum_{j=1}^{K} N_j^1} \qquad \hat{A}_{jk} = \frac{N_{jk}}{\sum_{k} N_{jk}}$$

HOW TO ESTIMATE ϕ:

① ASSUME MULTINOULLI EMISSION

$$\Rightarrow \phi_{jl} = p(x_t = l \mid z_t = j) \quad l \in \{1 \dots L\}$$

FROM MLE,

$$\boxed{\hat{\phi}_{jl} = \frac{N_{jl}^X}{N_j}} \; ; \quad N_{jl}^X \triangleq \sum_{i=1}^{N} \sum_{t=1}^{T_i} \mathbb{I}(z_{i,t} = j, x_{i,t} = l)$$

$\overline{\text{PVI}}$: HMM PARAMETER LEARNING GIVEN FULLY OBSERVED DATA

FROM MLE,

$$\hat{\pi}_j = \frac{N_j^1}{\sum_{j=1}^{K} N_j^1} \qquad \hat{A}_{jk} = \frac{N_{jk}}{\sum_k N_{jk}}$$

HOW TO ESTIMATE ϕ FOR NORMALLY DISTRIBUTED ϕ

$$\hat{\mu}_k = \frac{\bar{x}_k}{N_k} \qquad \hat{\Sigma}_k = \frac{\overline{(xx)}^T_k - N_k \hat{\mu}_k \hat{\mu}_k^T}{N_k}$$

$$\bar{x}_k \triangleq \sum_{i=1}^{N} \sum_{t=1}^{T_i} I(z_{i,t} = k)\, x_{i,t} \qquad \overline{(xx)}^T_k \triangleq \sum_{i=1}^{N} \sum_{t=1}^{T_i} I(z_{i,t} = k)\, x_{i,t} x_{i,t}^T$$

HMM LEARNING EXAMPLE (ESTIMATE π, A, Φ)

S_1	Bias	Fair	Fair	Bias
	H	T	H	H
S_2 :	F	F	F	F
	H	T	T	T
S_3 :	B	B	B	B
	T	H	H	H

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N_j^1 = # Times z_j occurs at $t=1$

$$= \sum_{i=1}^{N} I(z_{i1} = j)$$

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	T	H	H	H

N_j^1 = # Times z_j occurs at $t=1$

$$= \sum_{i=1}^{N} I(z_{i1} = j)$$

$N_B^1 = 2$ (S1 and S_3)

$N_F^1 = 1$ (S2)

HMM LEARNING EXAMPLE (ESTIMATE π, A, ϕ)

S_1	Bias	Fair	Fair	Bias
	H	T	H	H
S_2 :	F	F	F	F
	H	T	T	T
S_3 :	B	B	B	B
	T	H	H	H

$$N'_B = 2, \quad N'_F = 1$$

$$\hat{\pi} = [\hat{\pi}_B, \hat{\pi}_F] = \left[\frac{N'_B}{N'_B + N'_F}, \frac{N'_F}{N'_F + N'_B}\right] = \left[\frac{2}{3}, \frac{1}{3}\right]$$

HMM LEARNING EXAMPLE (ESTIMATE π, A, Φ)

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	H	T	H	H
S2 :	F	F	F	F
	H	T	T	T
S_3 :	B	B	B	B
	T	h	h	H

N_{jk} = COUNT OF TRANSITIONS FROM Z_j to Z_k.

HMM LEARNING EXAMPLE (ESTIMATE π, A, ϕ)

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	T	H	H	H

N_{jk} = COUNT OF TRANSITIONS FROM z_j to z_k.

$N_{BB} = 3$ $\quad$ $N_{FF} = 4$

$N_{BF} = 1$ $\quad$ $N_{FB} = 1$

HMM LEARNING EXAMPLE (ESTIMATE π, A, Φ)

S_1	Bias	Fair	fair	Bias
	H	T	H	H
S_2 :	F	F	F	F
	H	T	T	T
S_3 :	B	B	B	B
	T	H	H	H

$N_{BB} = 3$ $\qquad N_{FF} = 4$

$N_{BF} = 1$ $\qquad N_{FB} = 1$

$$\hat{A}_{BB} = \frac{N_{BB}}{N_{BB} + N_{BF}} = \frac{3}{4}$$

$$\hat{A}_{BF} = \frac{N_{BF}}{N_{BB} + N_{BF}} = \frac{1}{4}$$

$$\therefore \hat{A} = \begin{bmatrix} \hat{A}_{BB} & \hat{A}_{BF} \\ \hat{A}_{FB} & \hat{A}_{FF} \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} 0.75 & 0.25 \\ .2 & .8 \end{bmatrix}$$

HMM LEARNING EXAMPLE (ESTIMATE π, A, Φ)

S_1	B	F	F	B
	H	T	H	H
S_2 :	F	F	F	F
	H	T	T	T
S_3 :	B	B	B	B
	T	H	H	H

$$\hat{\Phi} = \begin{bmatrix} \hat{\Phi}_{BH} & \hat{\Phi}_{BT} \\ \hat{\Phi}_{FH} & \hat{\Phi}_{FT} \end{bmatrix}$$

HMM LEARNING EXAMPLE (ESTIMATE π, A, Φ)

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$$\hat{\Phi} = \begin{bmatrix} \hat{\Phi}_{BH} & \hat{\Phi}_{BT} \\ \hat{\Phi}_{FH} & \hat{\Phi}_{FT} \end{bmatrix}$$

$$\hat{\Phi}_{BH} = \frac{\text{COUNT of (Bias \& H)}}{\text{COUNT OF BIAS}}$$

HMM LEARNING EXAMPLE (ESTIMATE π, A, Φ)

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$$\hat{\Phi} = \begin{bmatrix} \hat{\Phi}_{BH} & \hat{\Phi}_{BT} \\ \hat{\Phi}_{FH} & \hat{\Phi}_{FT} \end{bmatrix}$$

$$\hat{\Phi}_{BH} = \frac{\text{COUNT of (Bias \& H)}}{\text{COUNT OF BIAS}}$$

$$= \frac{\text{COUNT }(B,H)}{\text{COUNT}(B,T) + \text{COUNT}(B,H)}$$

$$= \frac{4}{1 + 4}$$

$$= 0.8$$

$\hat{\Phi}_{BT} = 0.2$

$\hat{\Phi}_{FH} = 0.4$

$\hat{\Phi}_{FT} = 0.6$

PVII: HMM PARAMETER LEARNING WITHOUT FULLY OBSERVED DATA

* WITH MISSING DATA | LATENT VARIABLES (z_t) COMPUTING MLE | MAP IS HARD

* COULD USE GRADIENT BASED METHODS

PVII: HMM PARAMETER LEARNING WITHOUT FULLY OBSERVED DATA

* WITH MISSING DATA | LATENT VARIABLES (z_t) COMPUTING MLE | MAP IS HARD

* COULD USE GRADIENT BASED METHODS BUT TRICKY TO ENFORCE CONSTRAINTS like $\sum_j \pi_j = 1$ etc

* OFTEN USE EXPECTATION MAXIMISATION (EM)
  → ITERATIVE, CLOSED FORM UPDATES

EM ALGORITHM

Let x_i be observed variable
z_i be latent / hidden

{ General case
DON'T CONFUSE
NOTATION WITH HMM }

GOAL: MAXIMIZE LOG. LIKELIHOOD

$$l(\theta) = \sum_{i=1}^{N} \log p(x_i \mid \theta)$$

EM ALGORITHM

Let x_i be observed variable
z_i be latent / hidden

{ General case

DON'T CONFUSE NOTATION WITH HMM }

* GOAL: MAXIMIZE LOG. LIKELIHOOD

$$l(\theta) = \sum_{i=1}^{N} \log p(x_i \mid \theta) = \sum_{i=1}^{N} \log \left[\sum_{z_i} p(x_i, z_i \mid \theta) \right]$$

HARD TO SOLVE! (LOG CAN'T BE PUSHED INSIDE)

EM ALGORITHM

* GOAL: MAXIMIZE LOG. LIKELIHOOD

$$\ell(\theta) = \sum_{i=1}^{N} \log p(x_i | \theta) = \sum_{i=1}^{N} \log \left[\sum_{z_i} p(x_i, z_i | \theta) \right]$$

* DEFINE COMPLETE DATA LOG. LIKELIHOOD

$$\ell_c(\theta) \triangleq \sum_{i=1}^{N} \log p(x_i, z_i | \theta)$$

EM ALGORITHM

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* DEFINE COMPLETE DATA LOG. LIKELIHOOD

$$l_c(\theta) \triangleq \sum_{i=1}^{N} \log p(x_i, z_i \mid \theta)$$

$l_c(\theta)$ can not be computed $\because$ z_i is unknown

EM ALGORITHM

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$$\ell(\theta) = \sum_{i=1}^{N} \log p(x_i \mid \theta) = \sum_{i=1}^{N} \log \left[\sum_{z_i} p(x_i, z_i \mid \theta) \right]$$

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$$\ell_c(\theta) \triangleq \sum_{i=1}^{N} \log p(x_i, z_i \mid \theta)$$

* DEFINE EXPECTED COMPLETE DATA LIKELIHOOD

$$Q(\theta, \theta^{t-1}) = E\left[\ell_c(\theta) \mid D, \theta^{t-1} \right]$$

t: CURREN ITERATION

Q: AUXILIARY FUNCTION

EM ALGORITHM

* GOAL: MAXIMIZE LOG. LIKELIHOOD

$$\ell(\theta) = \sum_{i=1}^{N} \log p(x_i|\theta) = \sum_{i=1}^{N} \log \left[\sum_{z_i} p(x_i, z_i|\theta) \right]$$

* DEFINE COMPLETE DATA LOG. LIKELIHOOD

$$\ell_c(\theta) \triangleq \sum_{i=1}^{N} \log p(x_i, z_i|\theta)$$

* DEFINE EXPECTED COMPLETE DATA LIKELIHOOD

$$Q(\theta, \theta^{t-1}) = E\left[\ell_c(\theta)|D, \theta^{t-1}\right]$$

* E-step : COMPUTE Q $\Rightarrow$ COMPUTE EXPECTED SUFFICIENT STATS (ESS)

EM ALGORITHM

* GOAL: MAXIMIZE LOG. LIKELIHOOD

$$\ell(\theta) = \sum_{i=1}^{N} \log p(x_i \mid \theta) = \sum_{i=1}^{N} \log \left[\sum_{z_i} p(x_i, z_i \mid \theta) \right]$$

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* DEFINE EXPECTED COMPLETE DATA LIKELIHOOD

$$Q(\theta, \theta^{t-1}) = E\left[\ell_c(\theta) \mid D, \theta^{t-1} \right]$$

* E-step : COMPUTE Q
* M-step : OPTIMIZE Q wrt θ , i.e. $\theta^{t} = \underset{\theta}{\operatorname{argmax}} \, (\theta, \theta^{t-1})$

EM ALGORITHM (WITHOUT THE DERIVATION)

TWO STEPS (TILL CONVERGENCE)

① E : INFERRING MISSING VALUES $(Z_{i,t})$ GIVEN MODEL PARAMETERS $(\theta = (\pi, A, \phi))$

EM ALGORITHM

TWO STEPS (TILL CONVERGENCE)

① E : INFERRING MISSING VALUES $(Z_{i,t})$
GIVEN MODEL PARAMETERS $(\theta = \{\pi, A, \phi\})$

② M : OPTIMIZING PARAMETERS $(\theta = \{\pi, A, \phi\})$
USING MLE AND
FILLED IN DATA $(Z_{i,t})$

COMPUTING EXPECTED SUFFICIENT STATS (ESS)
(OR TERMS FROM AUX. FUNCTION ON WHICH MLE DEPENDS ON)

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(OR TERMS FROM AUX. FUNCTION ON
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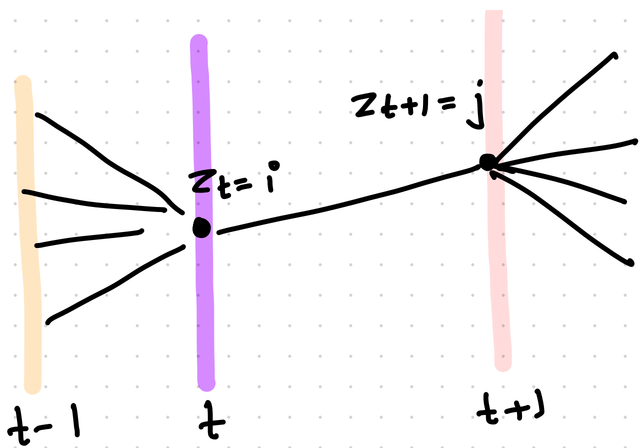
$$\xi_t(i,j) = P(z_t = i, z_{t+1} = j \mid x_{1:T}, \theta)$$

Probability of being in state 'i' at
't' and state 'j' at 't+1' given
model and observations

COMPUTING EXPECTED SUFFICIENT STATS (ESS)

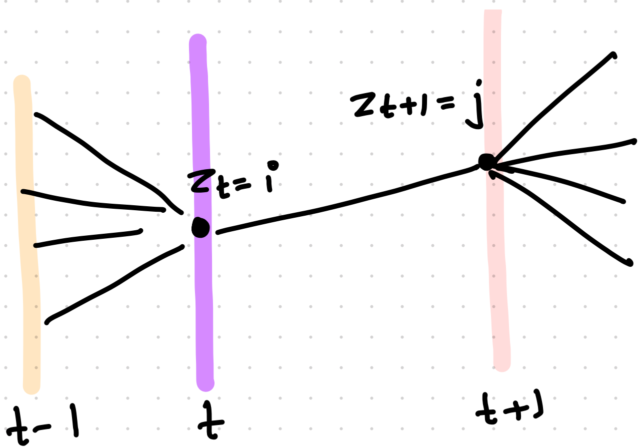
(OR TERMS FROM AUX. FUNCTION ON WHICH MLE DEPENDS ON)

$$\xi_t(i,j) = P(z_t = i, z_{t+1} = j \mid x_{1:T}, \theta)$$



$$\xi_t(i,j) = p(z_t = i, z_{t+1} = j \mid x_{1:T}, \theta)$$

$$= \frac{p(z_t = i, z_{t+1} = j, x_{1:T} \mid \theta)}{p(x_{1:T} \mid \theta)}$$



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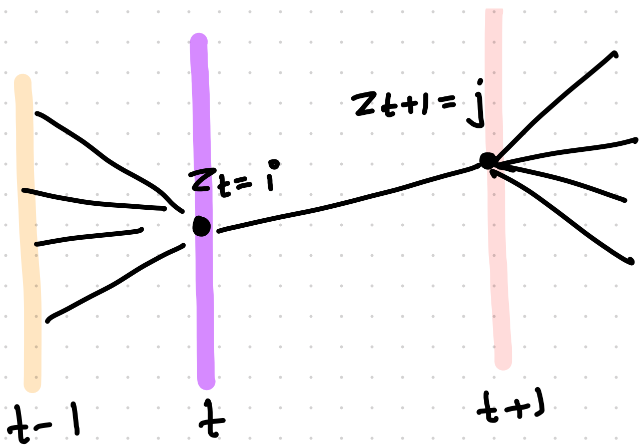
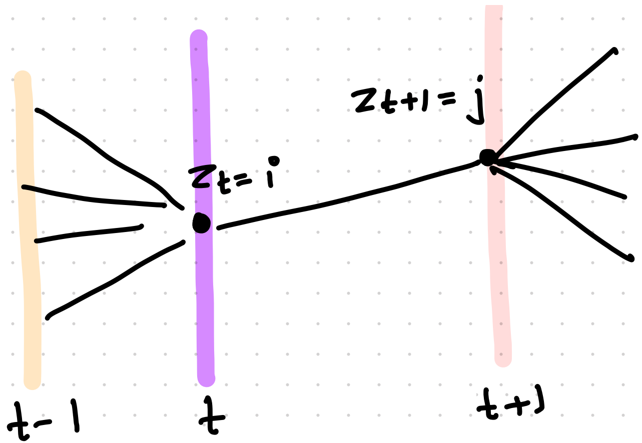


$$\xi_t(i,j) = \frac{P(z_t = i, x_{1:t})\, A_{ij}\, P(x_{t+1} \mid z_{t+1} = j)\, P(x_{t+1:T} \mid z_{t+1} = j)}{P(x_{1:T} \mid \theta)}$$

$$\xi_t(i,j) = P(z_t = i, z_{t+1} = j \mid x_{1:T}, \theta)$$



$$\xi_t(i,j) = \frac{P(z_t = i, x_{1:t})\, A_{ij}\, P(x_{t+1} \mid z_{t+1} = j)\, P(x_{t+1:T} \mid z_{t+1} = j)}{P(x_{1:T} \mid \theta)}$$

$$= \frac{\alpha_t(i)\, \phi_j(x_{t+1})\, \beta_{t+1}(j)\, A_{ij}}{\sum_i \sum_j \alpha_t(i)\, \phi_j(x_{t+1})\, \beta_t(j)\, A_{ij}}$$

$$\xi_t(i,j) = P(z_t = i, z_{t+1} = j \mid x_{1:T}, \theta)$$

NOW; $\gamma_t(i)$ = Probability of being in state 'i' at time 't' given observations and model.

$$\xi_t(i,j) = P(z_t = i, z_{t+1} = j \mid x_{1:T}, \theta)$$

Now; $\gamma_t(i) =$ Probability of being in state 'i' at time 't' given observations and model.

$$\Rightarrow \gamma_t(i) = \sum_{j=1}^{K} \xi_t(i,j)$$

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Q: What does $\sum_t \gamma_t(i)$ mean?

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Expected # of visits to state i

$\xi_t(i,j) = P(z_t = i, z_{t+1} = j \mid x_{1:T}, \theta)$

Now; $\gamma_t(i) =$ Probability of being in state 'i' at time 't' given observations and model.

$$\Rightarrow \gamma_t(i) = \sum_{j=1}^{K} \xi_t(i,j)$$

Q: What does $\sum_{t=1}^{T-1} \gamma_t(i)$ mean

Expected # of visits to state i

Or, Expected # of transitions from state i (ignoring $t = T$)

$$\xi_t(i,j) = P(z_t = i, z_{t+1} = j \mid x_{1:T}, \theta)$$

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$$\xi_t(i,j) = P(z_t = i, z_{t+1} = j \mid x_{1:T}, \theta)$$

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Expected # of visits to state i

Or, Expected # of transitions from state i (ignoring $t = T$)

Q: What does $\sum_{t=1}^{T-1} \xi_t(i,j)$ mean?

Expected # of transitions from state i to j.

HMM LEARNING EXAMPLE (ESTIMATE π, A, ϕ)

S_1	Bias	Fair	fair	Bias
	H	T	H	H
S2 :	F	F	F	F
	H	T	T	T
S_3 :	B	B	B	B
	T	H	H	H

Unknown

Let us assume:

← 1^{st} iteration (RANDOM INIT.)

$$\pi^1 = [0.9 \quad 0.1] \qquad A^1 = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

$$\phi_B^1 = \begin{array}{c} \begin{array}{cc} H & T \end{array} \\ \begin{bmatrix} 0.8 & 0.2 \end{bmatrix} \end{array} \qquad \phi_F^1 = \begin{array}{c} \begin{array}{cc} H & T \end{array} \\ \begin{bmatrix} 0.4 & 0.6 \end{bmatrix} \end{array}$$

HMM LEARNING EXAMPLE (ESTIMATE π, A, ϕ)

CONSIDER S1 ONLY FOR NOW

S_1

H T H H

$\pi^1 = [0.9 \quad 0.1]$

$A^1 = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$

$\phi_B^1 = \begin{array}{cc} H & T \\ \left[0.8 \right. & \left. 0.2 \right] \end{array}$ $\quad \phi_F^1 = \begin{array}{cc} H & T \\ \left[0.4 \right. & \left. 0.6 \right] \end{array}$

$\eta_1(B, B) = ?$ $\quad \eta_1(B, F) = ?$ $\quad \eta_1(F, F) = ?$ $\quad \eta_1(F, B) = ?$

HMM LEARNING EXAMPLE (ESTIMATE π, A, ϕ)

CONSIDER S1 ONLY FOR NOW

S_1 H T H H

$$\pi^1 = [0.9 \quad 0.1] \qquad A^1 = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

$$\phi^1_B = \begin{array}{c} \begin{array}{cc} H & T \end{array} \\ [0.8 \quad 0.2] \end{array} \qquad \phi^1_F = \begin{array}{c} \begin{array}{cc} H & T \end{array} \\ [0.4 \quad 0.6] \end{array}$$

$$\xi_1(B,B) = \frac{\alpha_1(B) . A_{BB}\, \phi_B(x_2 = T) . \beta_2(B)}{\text{NORMALIZATION CONSTANT}}$$

HMM LEARNING EXAMPLE (ESTIMATE π, A, ϕ)

CONSIDER S1 ONLY FOR NOW

S_1	H	T	H	H

$$\pi^1 = [0.9 \quad 0.1] \qquad A^1 = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

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$$\xi_1(B,B) = \frac{\alpha_1(B) \cdot A_{BB}\, \phi_B(x_2 = T) \cdot \beta_2(B)}{\text{NORMALIZATION CONSTANT}}$$

$$\alpha_1(B) = \pi_B\, \phi_B(x_1 = H) = (0.9)(0.8) = 0.72$$

$$\beta_2(B) = ?$$

HMM LEARNING EXAMPLE (ESTIMATE π, A, ϕ)

CONSIDER S1 ONLY FOR NOW

S_1	H	T	H	H

$\pi' = [0.9 \quad 0.1]$ $A' = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$

$\phi'_B = \begin{array}{cc} H & T \\ \end{array} [0.8 \quad 0.2]$ $\phi'_F = \begin{array}{cc} H & T \\ \end{array} [0.4 \quad 0.6]$

$\beta_4(B) = 1$ & $\beta_4(F) = 1$

$\beta_3(B) = A_{BB}\,\phi_B(x_4 = H)\,\beta_4(B) + A_{BF}\,\phi_F(x_4 = H)\,\beta_4(F)$

$= (0.7)(0.8) + (0.3)(0.4) = .56 + .12 = 0.68$

HMM LEARNING EXAMPLE (ESTIMATE π, A, ϕ)

CONSIDER S1 ONLY FOR NOW

S_1	H	T	H	H

$\pi^1 = [0.9 \quad 0.1]$ $\qquad A^1 = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$

$\phi_B^1 = \begin{array}{cc} H & T \\ \hline 0.8 & 0.2 \end{array}$ $\qquad \phi_F^1 = \begin{array}{cc} H & T \\ \hline 0.4 & 0.6 \end{array}$

$\beta_4(B) = 1 \quad \& \quad \beta_4(F) = 1$

$\beta_3(B) = A_{BB}\,\phi_B(x_4 = H)\,\beta_4(B) + A_{BF}\,\phi_F(x_4 = H)\,\beta_4(F)$

$= (0.7)(0.8) + (0.3)(0.4) = .56 + .12 = 0.68$

$\beta_2(B) = A_{BB}\,\phi_B(x_3 = H)\,\beta_3(B) + A_{BF}\,\phi_F(x_3 = H)\,\beta_3(F)$

$= (0.7)(0.8)(0.68) + (0.3)(0.4)(1 - .68) = 0.4192$

HMM LEARNING EXAMPLE (ESTIMATE π, A, ϕ)

CONSIDER S1 ONLY FOR NOW

S_1	H	T	H	H

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$$\xi_1(B,B) = \frac{\alpha_1(B) \cdot A_{BB}\, \phi_B(x_2 = T) \cdot \beta_2(B)}{\text{NORMALIZATION CONSTANT } (C)}$$

$$= \frac{0.72 * 0.7 * 0.2 * .4192}{C}$$

HMM LEARNING EXAMPLE (ESTIMATE π, A, ϕ)

CONSIDER S1 ONLY FOR NOW

S_1 H T H H

$\pi^1 = [0.9 \quad 0.1]$

$A^1 = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$

$\phi_B^1 = \begin{array}{cc} H & T \\ \end{array}$ $[0.8 \quad 0.2]$

$\phi_F^1 = \begin{array}{cc} H & T \\ \end{array}$ $[0.4 \quad 0.6]$

We can similarly find $\eta_1(B,F)$, $\eta_1(F,F)$, $\eta_1(F,B)$ for this sequence

HMM LEARNING EXAMPLE (ESTIMATE π, A, Φ)

CONSIDER S1 ONLY FOR NOW

S_1	H	T	H	H

$$\pi^1 = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix} \qquad A^1 = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

$$\Phi^1_B = \begin{array}{c} \begin{array}{cc} H & T \end{array} \\ \begin{bmatrix} 0.8 & 0.2 \end{bmatrix} \end{array} \qquad \Phi^1_F = \begin{array}{c} \begin{array}{cc} H & T \end{array} \\ \begin{bmatrix} 0.4 & 0.6 \end{bmatrix} \end{array}$$

we can find.

$$\gamma_1(B) = \xi_1(B, F) + \xi_1(B, B)$$

$$\gamma_1(F) = 1 - \gamma_1(B)$$

HMM LEARNING EXAMPLE (ESTIMATE π, A, Φ)

S_1

H T H H

CONSIDER ONLY THE STATES AT $t=1$

$$P(Z_1 = B) = \gamma_1(B)$$

$$P(Z_1 = F) = \gamma_1(F)$$

HMM LEARNING EXAMPLE (ESTIMATE π, A, Φ)

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H T H H

CONSIDER ONLY THE STATES AT $t=1$

$$P(Z_1 = B) = \gamma_1(B)$$

$$P(Z_1 = F) = \gamma_1(F)$$

we don't know for sure whether Z_1 is B or F, but we know the probabilies.

KEY INSIGHT: CONSIDER $\gamma_1(j)$ = EXPECTED # OF TIMES $Z_1 = j$

HMM LEARNING EXAMPLE (ESTIMATE π, A, ϕ)

KEY INSIGHT: CONSIDER $\gamma_1(j)$ = EXPECTED # OF TIMES $z_1 = j$

NOW AS WE DID FOR FULLY OBSERVED CASE, WE CAN USE MLE to estimate

π^2, A^2, ϕ^2

EM ALGORITHM

TWO STEPS (TILL CONVERGENCE)

① E : INFERRING MISSING VALUES $(z_{i,t})$
GIVEN MODEL PARAMETERS $(\theta = \{\pi, A, \phi\})$

COMPUTE ESS

$$\gamma_{i,t}(j) \triangleq p(z_t = j \mid x_{i,1:T_i}, \theta)$$

$$\xi_{i,t}(j,k) \triangleq p(z_{t-1} = j, z_t = k \mid x_{i,1:T_i}, \theta)$$

EM ALGORITHM

TWO STEPS (TILL CONVERGENCE)

② M : OPTIMIZING PARAMETERS ($\theta = \{\pi, A, \phi\}$)

USING MLE AND

FILLED IN DATA ($z_{i,t}$) OR <u>ESS</u>

$$\hat{\pi}_k = \text{Expected Freq. of } z_{i,t} = \text{state } k$$

$$= \frac{\sum_{i=1}^{N} \gamma_{i,1}(k)}{N}$$

EM ALGORITHM

TWO STEPS (TILL CONVERGENCE)

② M : OPTIMIZING PARAMETERS ($\theta = \{\pi, A, \phi\}$)

USING MLE AND

FILLED IN DATA ($z_{i,t}$) OR ESS

$$\hat{A}_{jk} = \frac{\text{Expected \# transitions } z_t = j \text{ to } z_{t+1} = k}{\text{Expected \# transitions from } z_t = j}$$

EM ALGORITHM

TWO STEPS (TILL CONVERGENCE)

② M : OPTIMIZING PARAMETERS ($\theta = \{\pi, A, \phi\}$)

USING MLE AND

FILLED IN DATA ($z_{i,t}$) OR <u>ESS</u>

$$\hat{A}_{jk} = \frac{\text{Expected \# transitions } z_t = j \text{ to } z_{t+1} = k}{\text{Expected \# transitions from } z_t = j}$$

$$= \frac{\sum_{i=1}^{N} \sum_{t=2}^{T_i} \eta_{i,t}(j,k)}{\sum_{k=1}^{K} \sum_{i=1}^{N} \sum_{t=2}^{T_i} \eta_{i,t}(j,k)} \leftarrow \text{NORMALIZATION}$$

EM ALGORITHM

TWO STEPS (TILL CONVERGENCE)

② M : OPTIMIZING PARAMETERS ($\theta = \{\pi, A, \phi\}$)

USING MLE AND

FILLED IN DATA ($z_{i,t}$) OR ESS

$$\hat{\phi}_{jl} = \frac{\text{Expected \# times in state 'j' and observing 'l'}}{\text{Expected \# times in state j}}$$

$$= \frac{\sum_{i=1}^{N} \sum_{t: x_{i,t} = l} \gamma_{i,t}(j)}{\sum_{i=1}^{N} \sum_{t=1}^{T_i} \gamma_{i,t}(j)}$$

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   - KNOWN Zs

3. INIT. USING VITERBI TRAINING