Derivation of Bivariate Normal Distribution

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Univariate Normal

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Two Independent Normals

Let:

$$X \sim \mathcal{N}(\mu_1, \sigma_1^2), \quad Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$$

Assuming independence:

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(y-\mu_2)^2}{2\sigma_2^2}\right)$$

Vector Notation

Define:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

Covariance matrix (independent case):

$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

Determinant of Σ

Since diagonal:

$$|\Sigma| = \sigma_1^2 \cdot \sigma_2^2$$

Inverse of Σ

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 \\ 0 & \frac{1}{\sigma_2^2} \end{bmatrix}$$

Mahalanobis Term

Let:

$$\mathbf{d} = \mathbf{x} - \boldsymbol{\mu} = \begin{bmatrix} x - \mu_1 \\ y - \mu_2 \end{bmatrix}$$

Then:

$$\mathbf{d}^\top \Sigma^{-1} \mathbf{d} = \frac{(x - \mu_1)^2}{\sigma_1^2} + \frac{(y - \mu_2)^2}{\sigma_2^2}$$

Putting it all together

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

Substitute:

$$= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{1}{2}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right)$$

Generalization: Correlated Case

If $\mathrm{Cov}(X, Y) = \rho\sigma_1\sigma_2$, then:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

Determinant:

$$|\Sigma| = \sigma_1^2\sigma_2^2(1 - \rho^2)$$

Bivariate Normal (Correlated)

$$f(\mathbf{x}) = \frac{1}{2\pi\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{x}-\boldsymbol{\mu})\right)$$

Valid for any symmetric, positive-definite Σ.