Integration of Joint PDF in a Region

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Joint PDF Definition

Let (X, Y) have a uniform joint PDF:

$$f_{X,Y}(x, y) = \frac{1}{4}, \quad 0 \leq x \leq 2, 0 \leq y \leq 2.$$

Otherwise, $f_{X,Y}(x, y) = 0$.

Defining the Region A

Consider the region:

$$A = \{(x, y) \mid x + y \leq 2\}$$

This forms a right triangle with vertices $(0, 0), (2, 0), (0, 2)$.

Computing $P(A)$

The probability is given by:

$$P(A) = \int_0^2 \int_0^{2-y} f_{X,Y}(x, y)\, dx\, dy.$$

Substituting $f_{X,Y}(x, y) = \frac{1}{4}$:

$$P(A) = \int_0^2 \int_0^{2-y} \frac{1}{4}\, dx\, dy.$$

Evaluating the Integral

Compute the inner integral:

$$\int_0^{2-y} \frac{1}{4}\, dx = \frac{1}{4}(2-y).$$

Compute the outer integral:

$$P(A) = \int_0^2 \frac{1}{4}(2-y)\, dy.$$

$$P(A) = \frac{1}{4}\left[2y - \frac{y^2}{2}\right]_0^2.$$

$$P(A) = \frac{1}{4} \times (4-2) = \frac{1}{4} \times 2 = \frac{1}{2}.$$

Conclusion

- The probability of A is $P(A) = \frac{1}{2}$.
- This matches the geometric interpretation: The area of the triangle is $\frac{1}{2} \times 2 \times 2 = 2$, and $P(A)$ follows from uniform probability.
- The integral visualization in GeoGebra/Desmos helps understand the computation.