Cumulative Distribution Function

(CDF)

Question) Given

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Generate samples from X.

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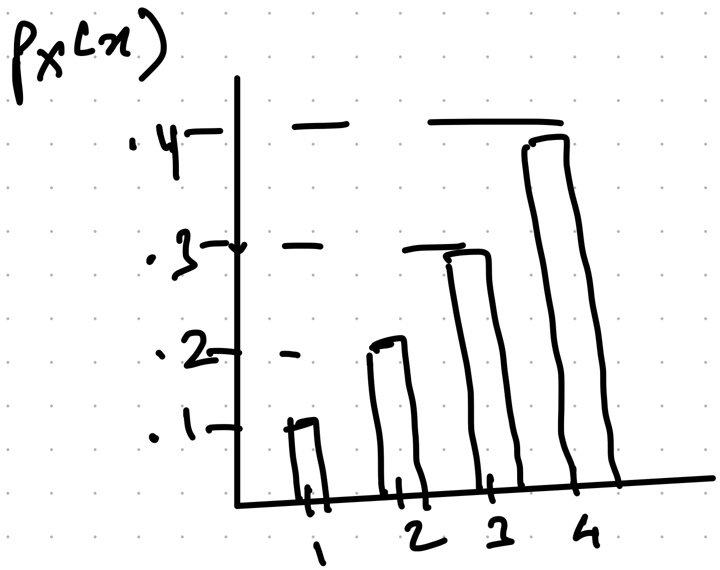

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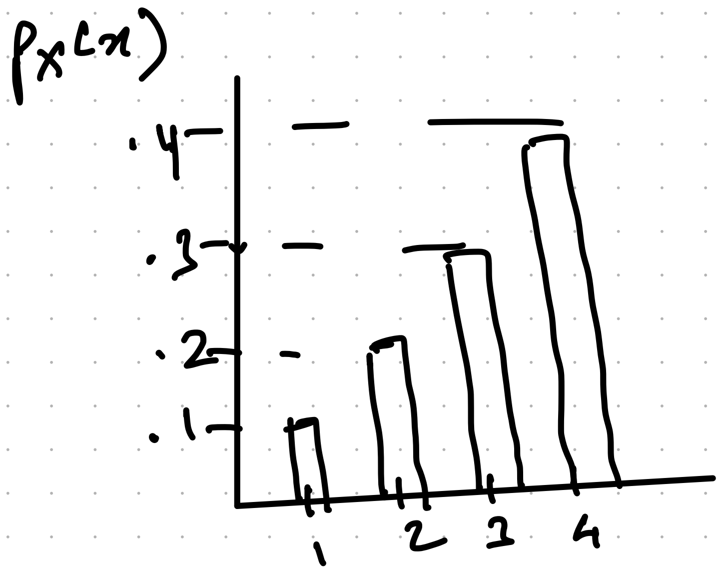

$\xrightarrow[\text{sum}]{\text{Cummulative}}$

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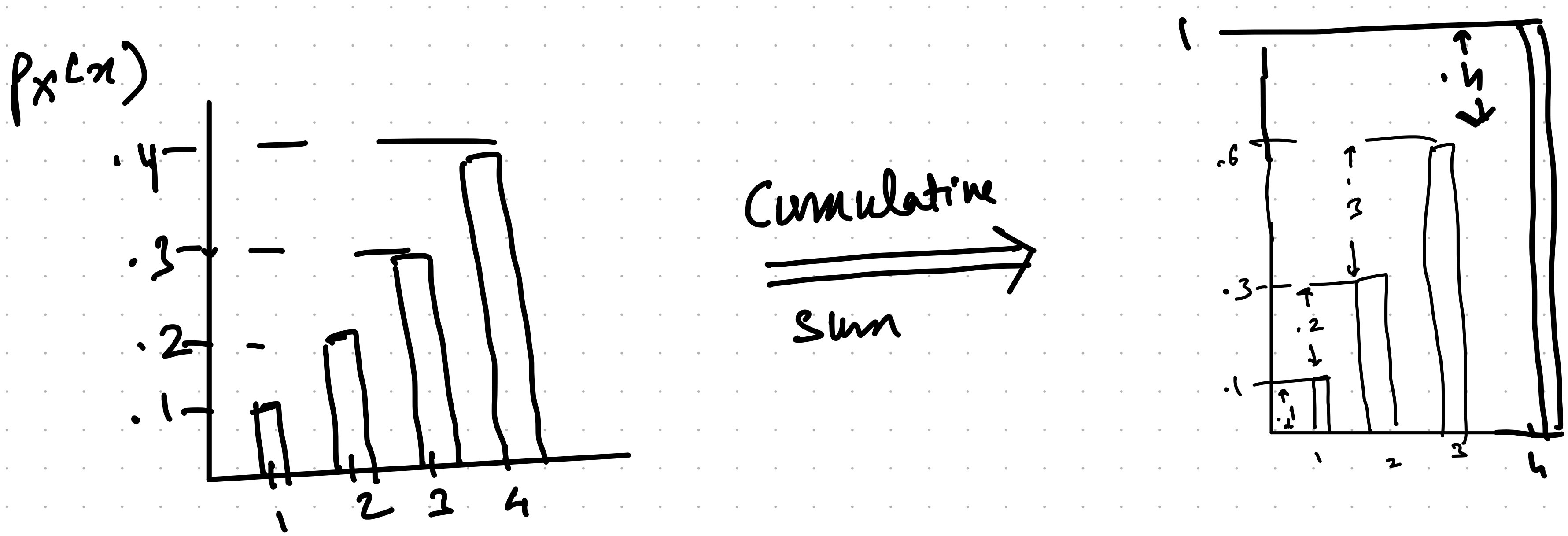
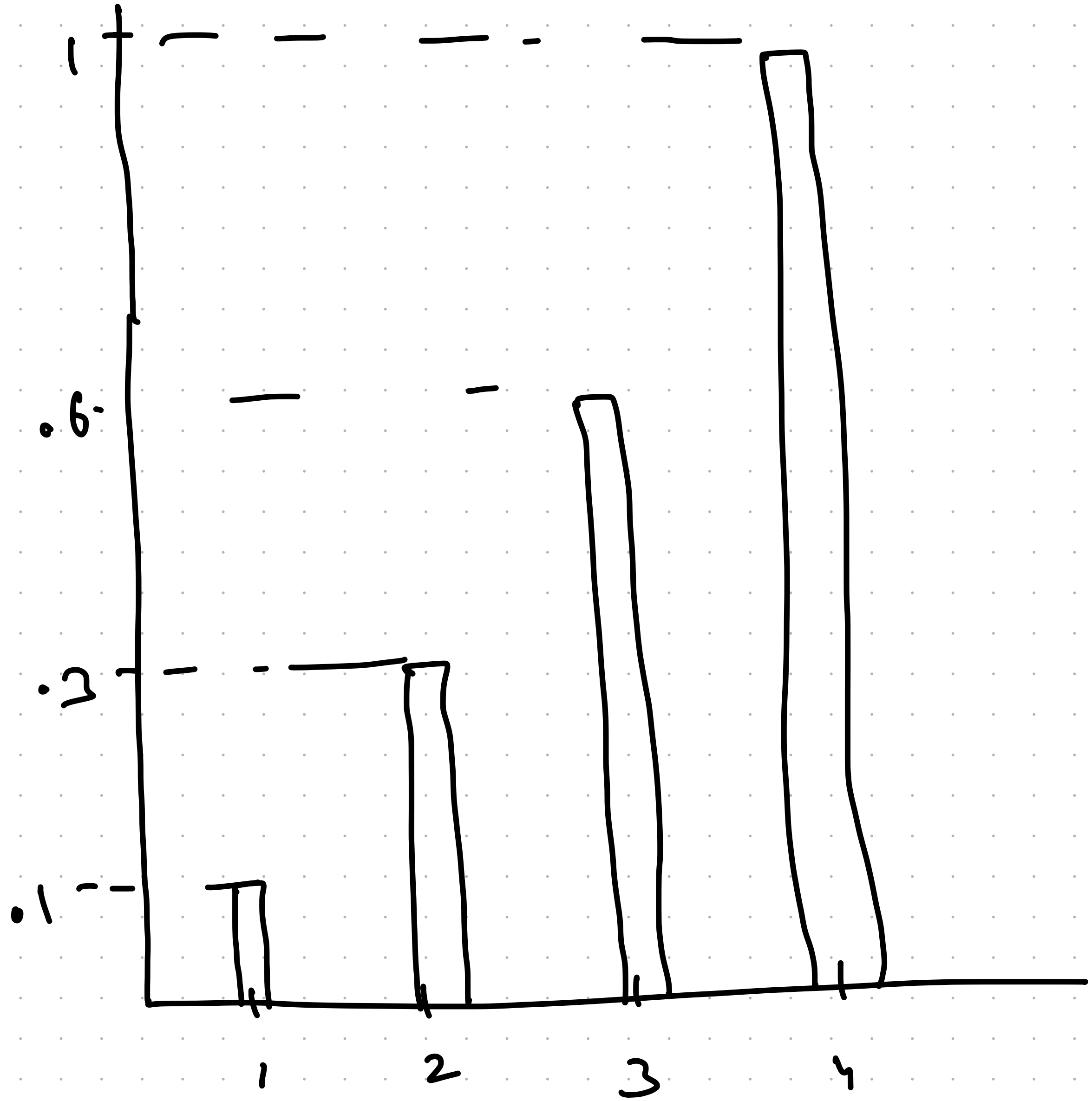
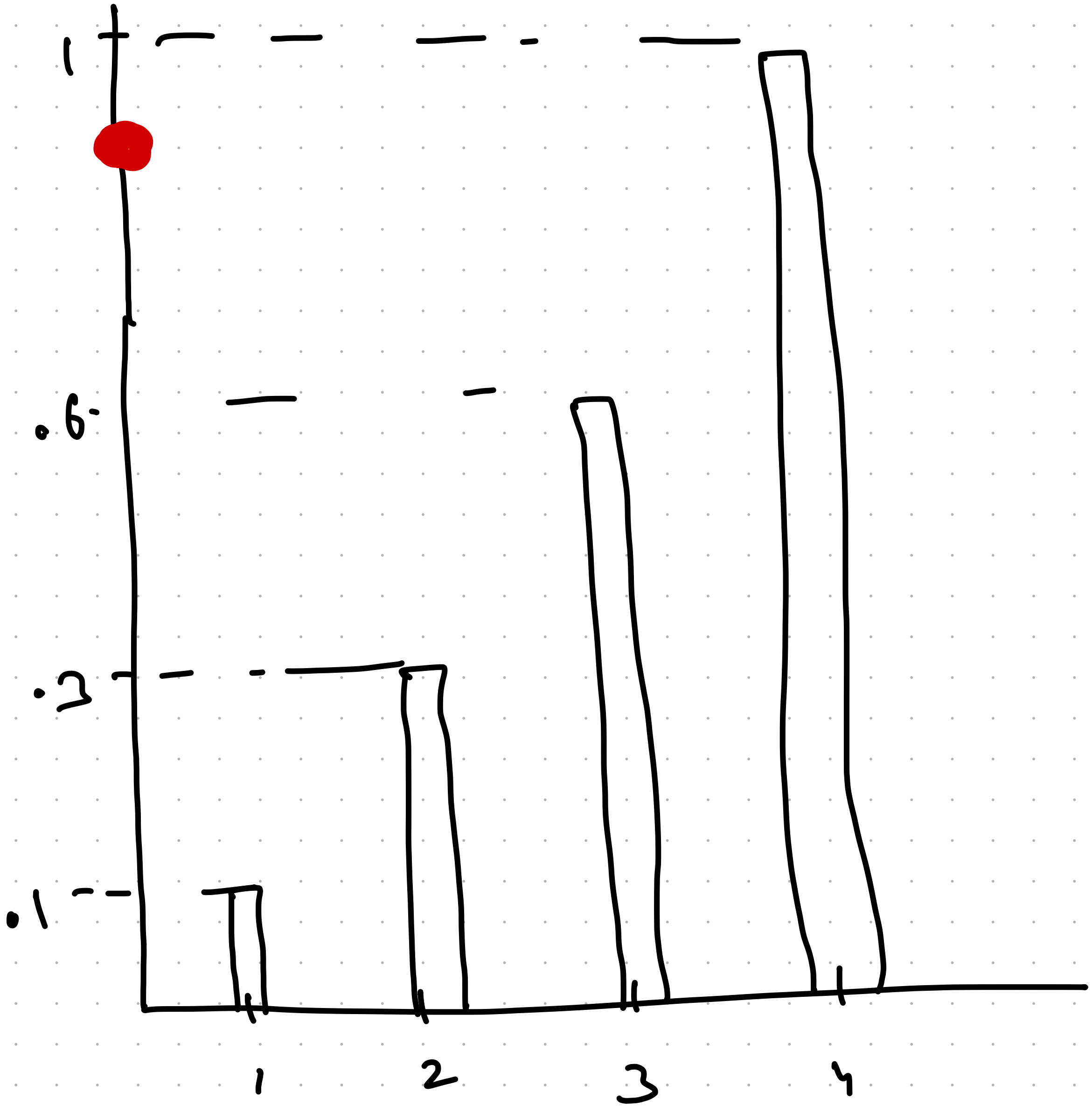
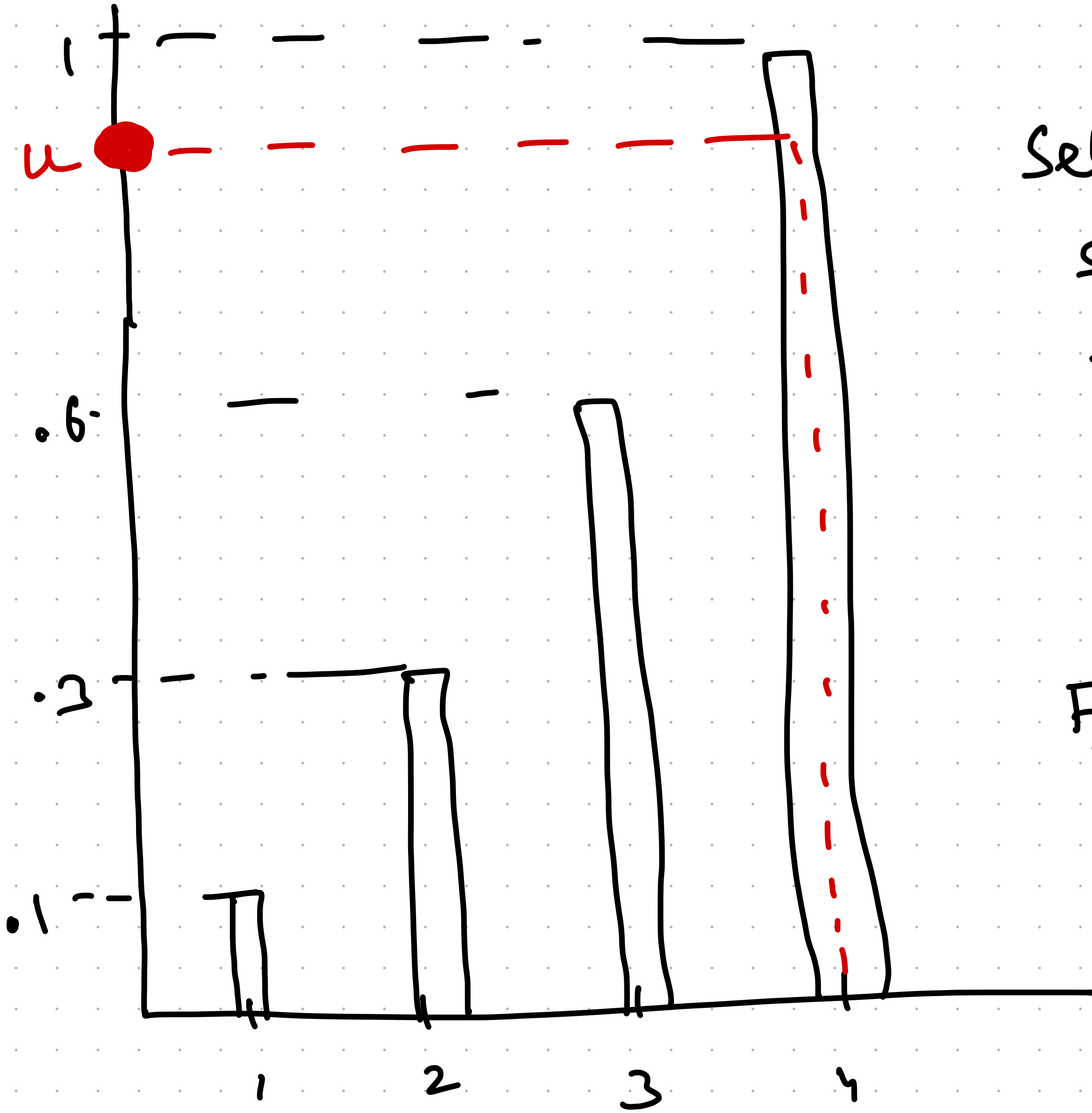




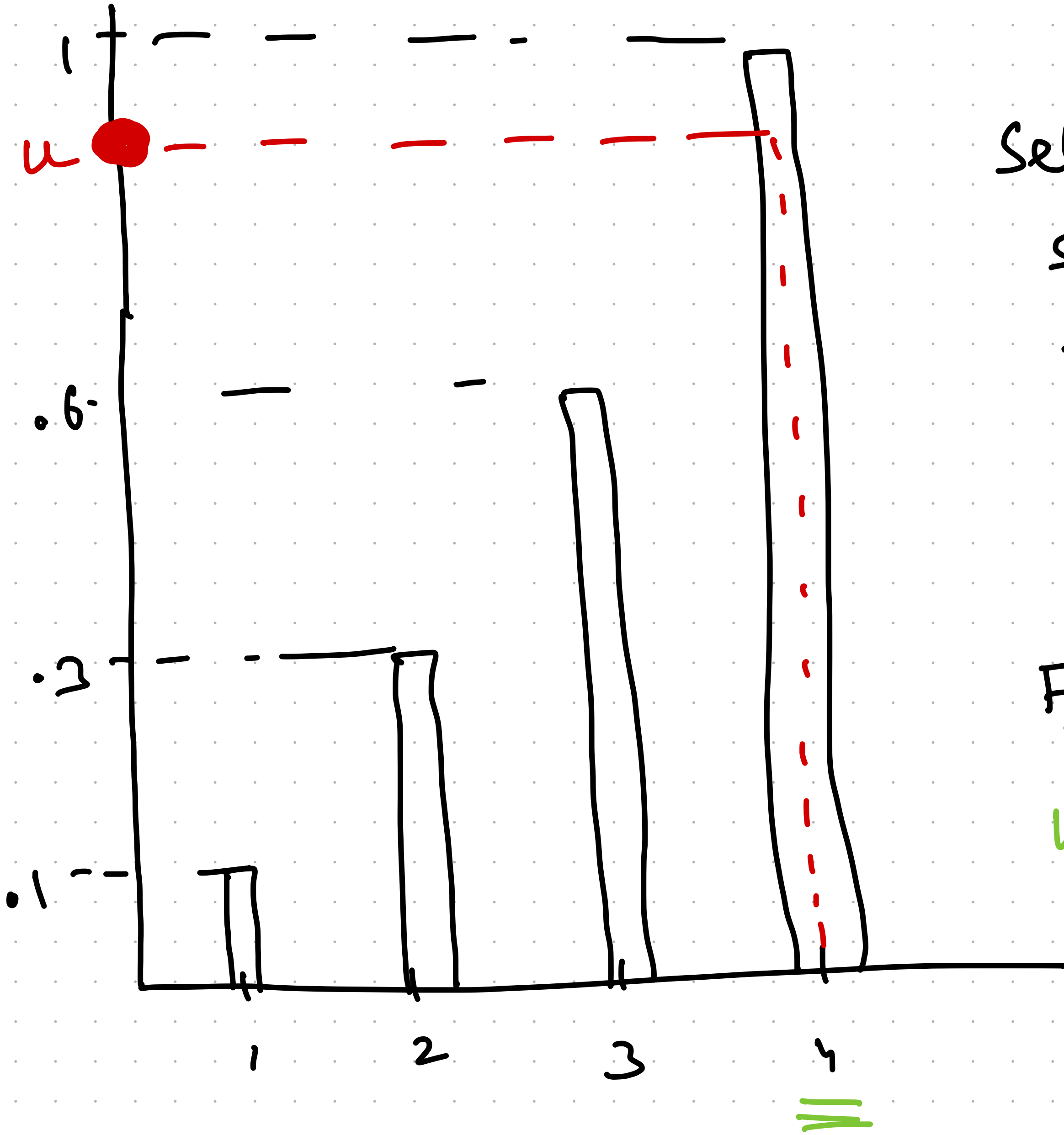
Cumulative sum of $p_X(x)$ given as $F_X(x)$



Sample from
U ~ Unif (0,1)
and put on
Y- axis



Select 'k'
s.t.
k is the smallest
x for
which
$F_X(x) \leq u$



Select 'k'

s.t.

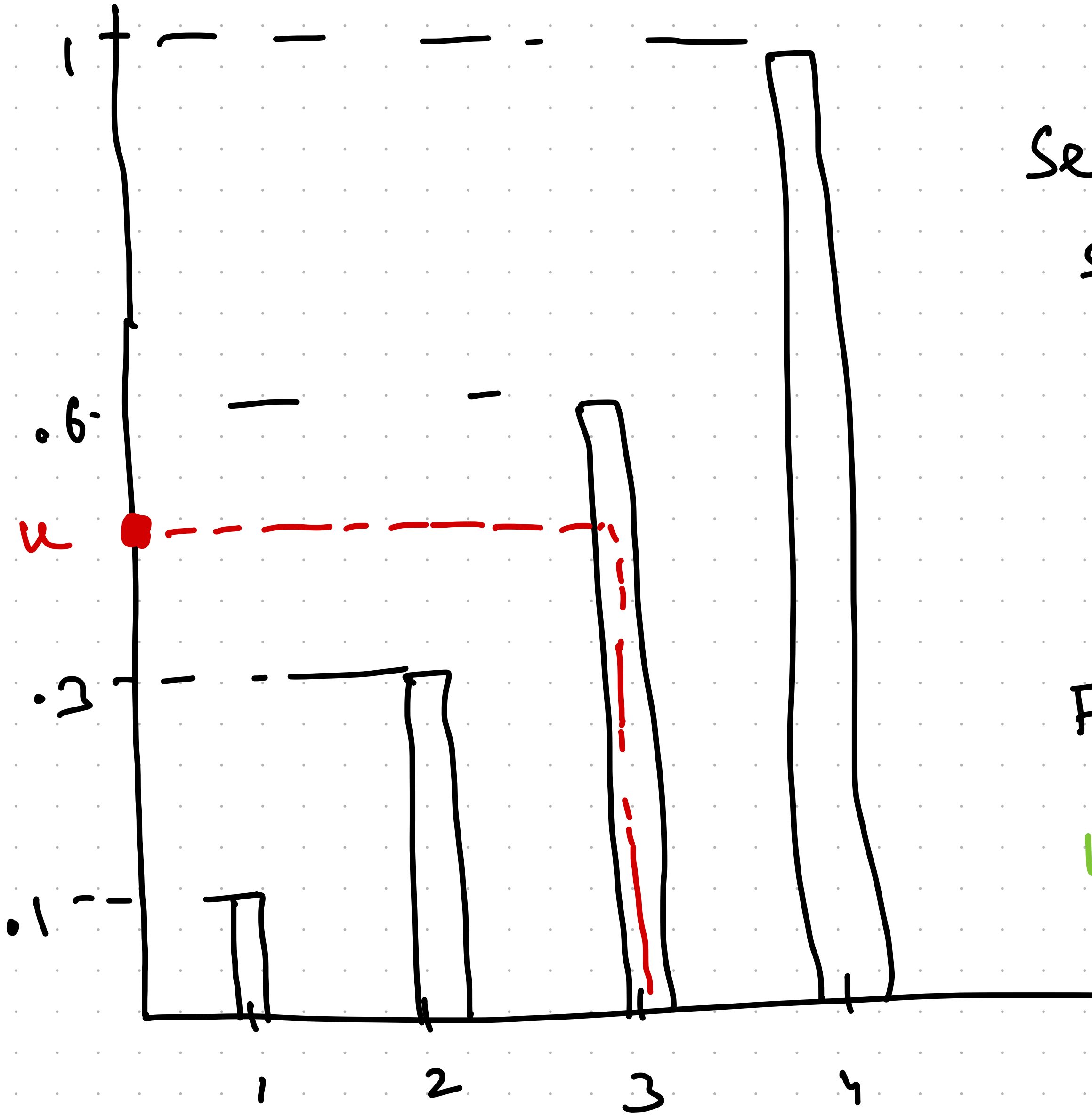
k is the smallest

x for

which

$F_X(x) \leq u$

we sample 'y'



Select 'k'
s.t.
k is the smallest
x for
which
$F_X(x) \leq u$

we sample
'3'

Notebook | Proof coming up

Background: Right and left continuous functions.

If $f(b) = f(b^-) \stackrel{def}{=} \lim_{h \to 0} f(b-h)$

then

f is left continuous at 'b'

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If $\quad f(b) = f(b^+) \overset{\text{def}}{=} \lim_{h \to 0} f(b+h)$

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f is right continuous at 'b'

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$$f(x) = \begin{cases} 1; & x < 1 \\ 2; & x \geq 1 \end{cases}$$

Is this funcⁿ left continuous or right continuous? at $x = 1$

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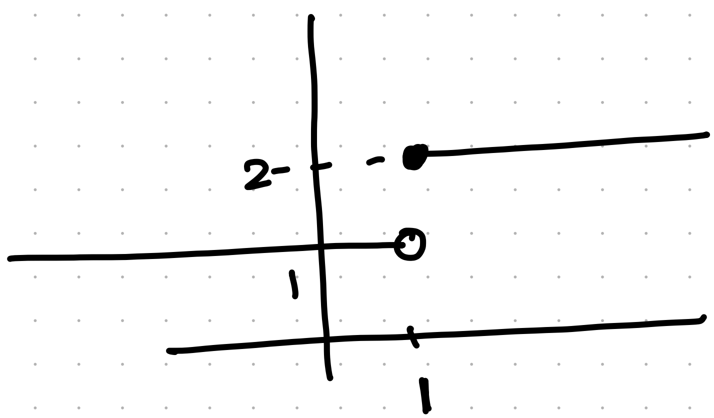

$f(1^+) = 2 = f(1)$

$f(1^-) = 1 \neq f(1)$

Right continuous

Say we have

$$f(x) = \begin{cases} 1; & x \leq 1 \\ 2; & x > 1 \end{cases}$$

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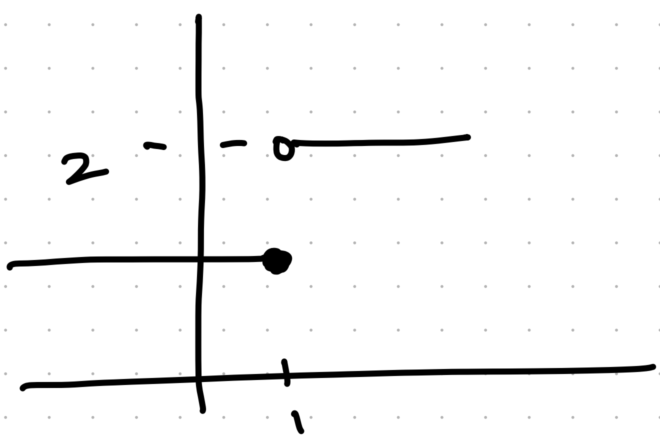

Left continuous

$f(1^-) = f(1) = 1$

$f(1^+) \neq f(1)$

CDF

CDF of discrete r.v. X is

$$F_X(x) \stackrel{\text{def}}{=} P[X \leq x] = \sum_{k \leq x} p_X(k)$$

* CDF is integration of PMF

Q) R.V. X has PMF

$p_X(0) = \frac{1}{4}$; $p_X(1) = \frac{1}{2}$; $p_X(4) = \frac{1}{4}$

Evaluate CDF

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Evaluate CDF

$$F_X(0) = P[X \le 0] = \frac{1}{4}$$

$$F_X(1) = P[X \le 1] = P[X=0] + P[X=1] = \frac{3}{4}$$

$$F_X(4) = P[X \le 4] = 1$$

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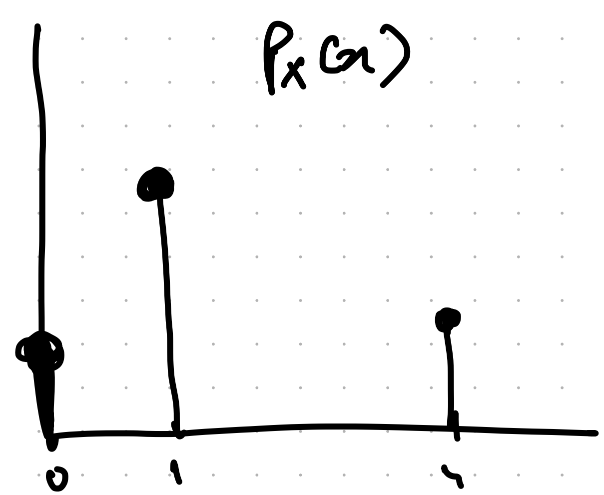
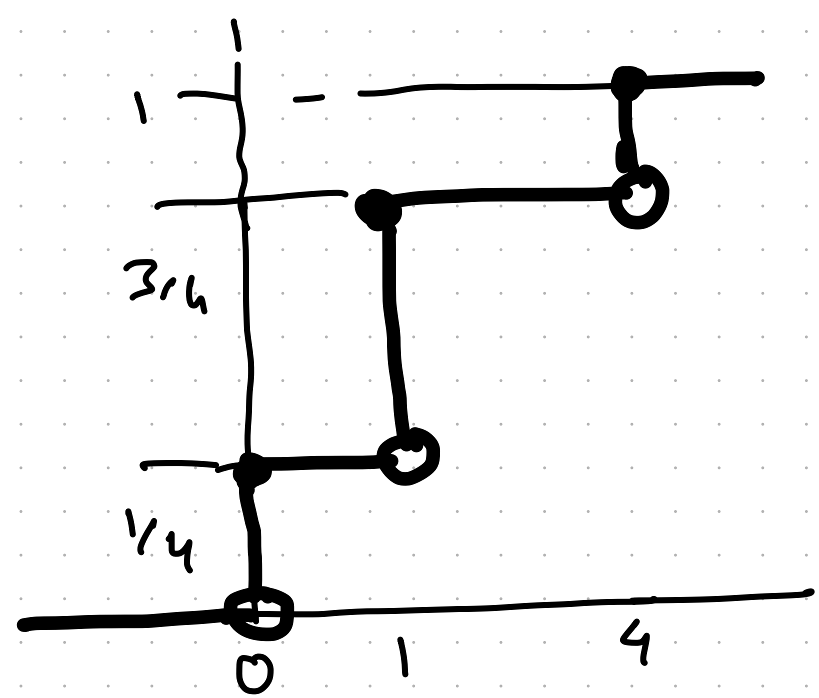


comsum $\longrightarrow$



Properties of CDF

① CDF is sequence of increasing steps

② $F_X(\infty) = 1$

③ $F_X(-\infty) = 0$

④ when $p_X(x) > 0$; there is a jump

⑤ Height of jump is $P(X = k)$

⑥ CDF is right continuous

CDF $\rightarrow$ PMF

Given r.v. X with CDF $F_X(x)$

find its PMF

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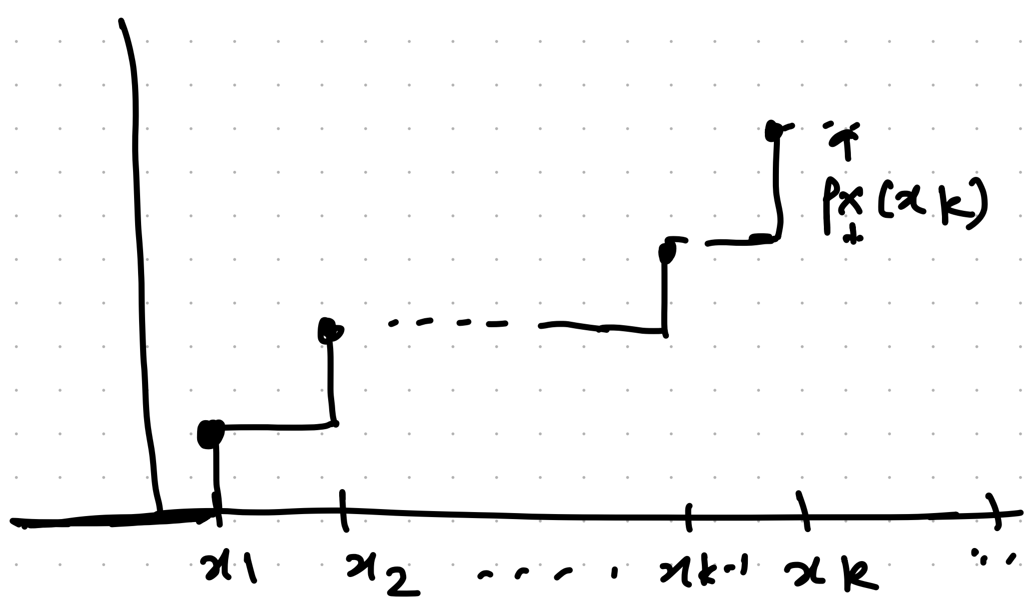
Given r.v. X with CDF $F_X(x)$

find its PMF

$$P_X(x_k) = F_X(x_k) - F_X(x_{k-1})$$



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Find PMF

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Find PMF

$$p_X(0) = \frac{1}{4} = F_X(0) - F_X(-\infty)$$

$$p_X(1) = F_X(1) - F_X(0) = \frac{1}{2}$$

$$p_X(4) = \frac{1}{4}$$

CDF for continuous r.v.

For continuous r.v. X

$$F_X(x) \stackrel{def}{=} P[X \leq x]$$

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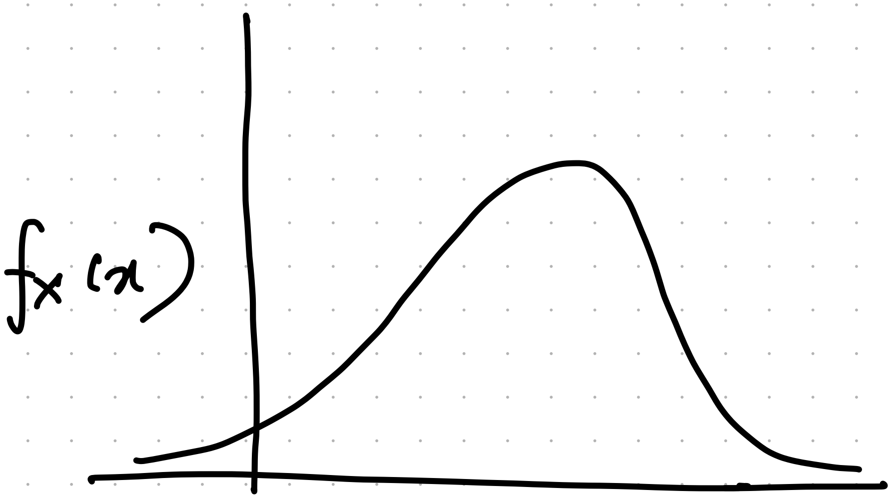

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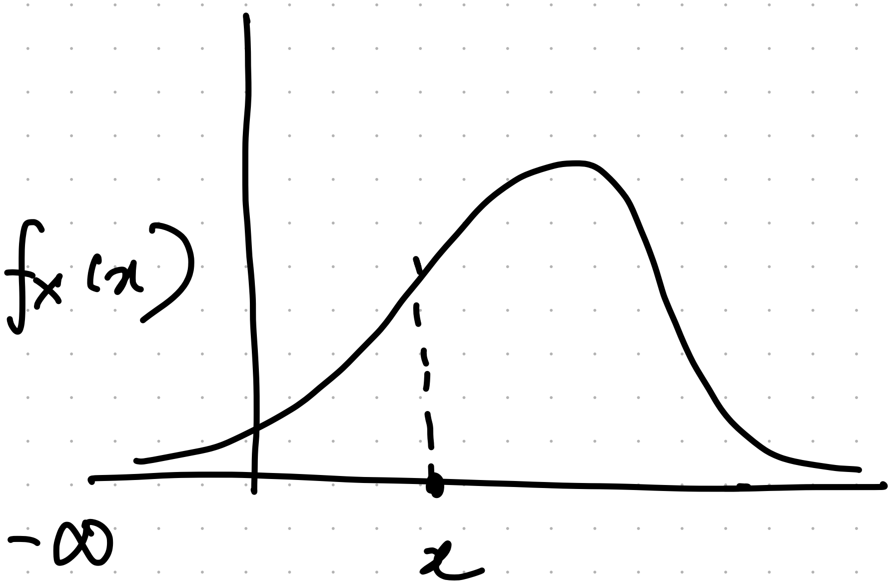

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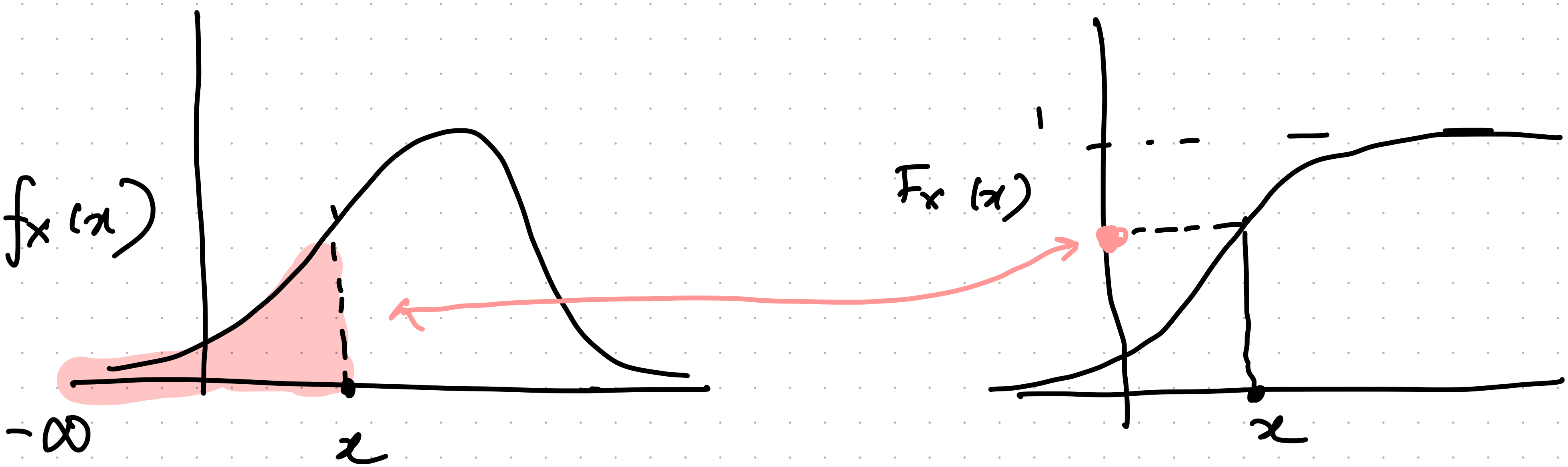

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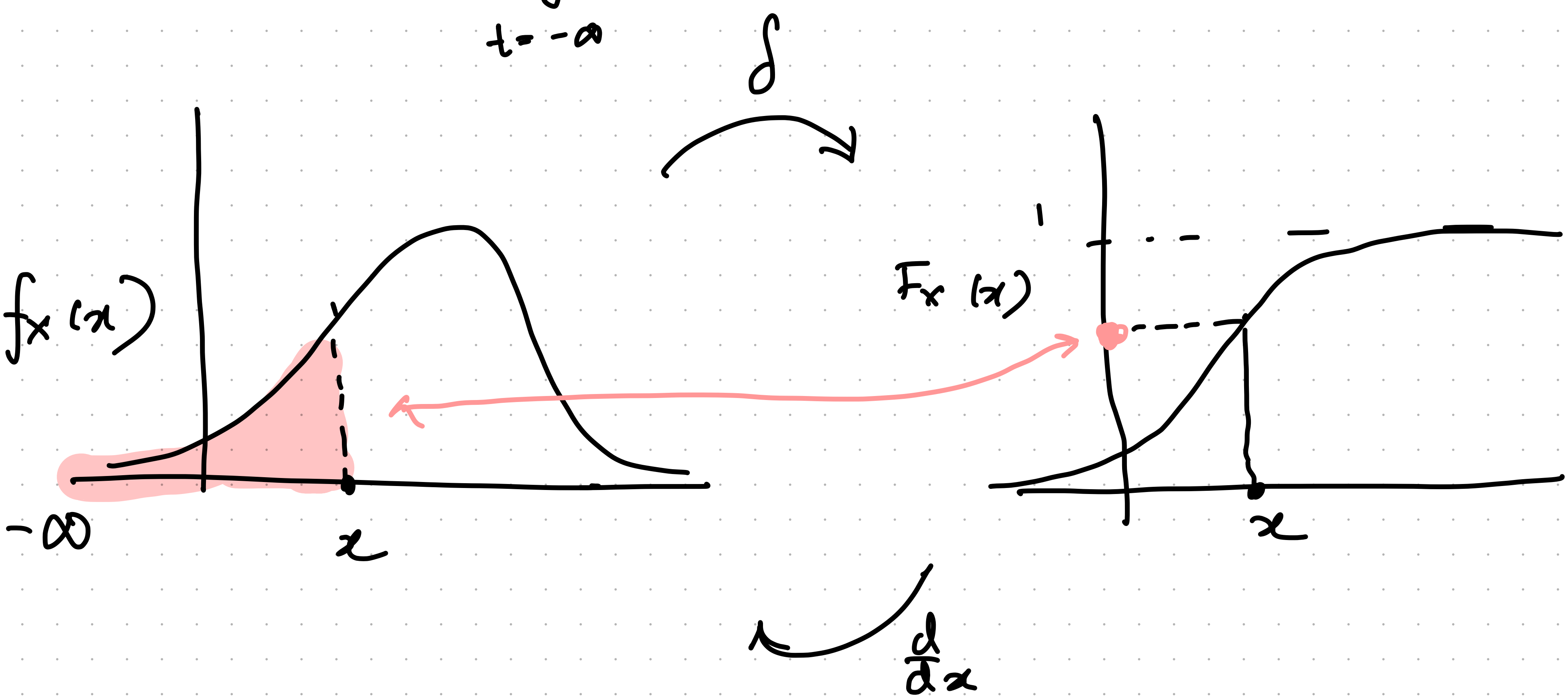

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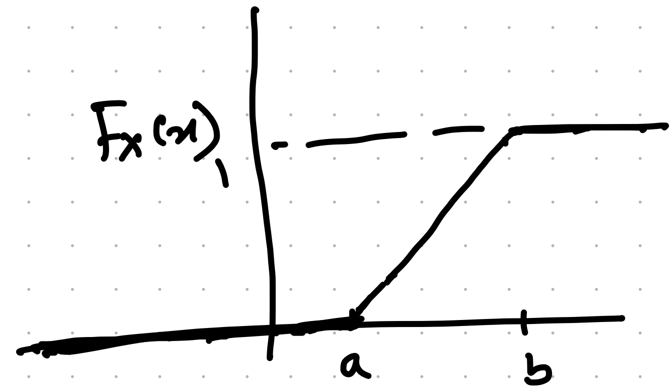

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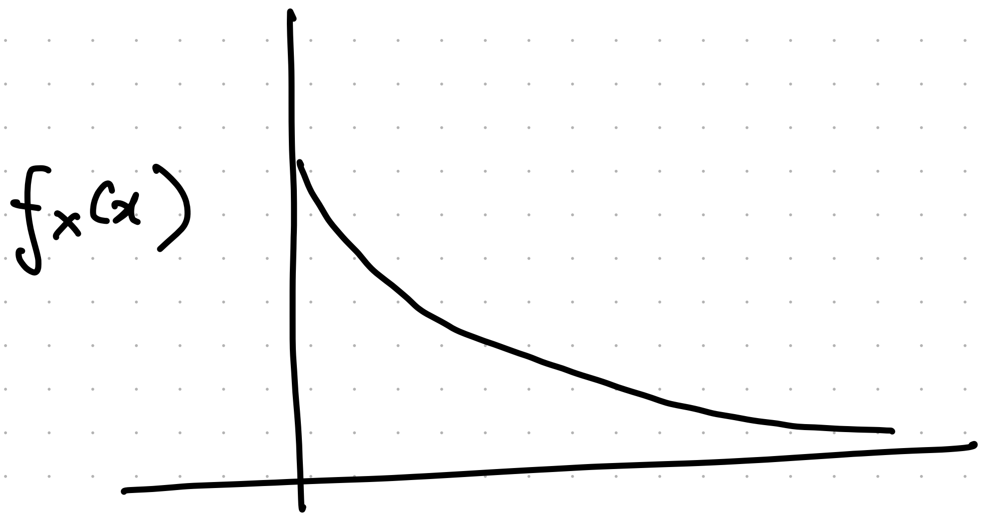

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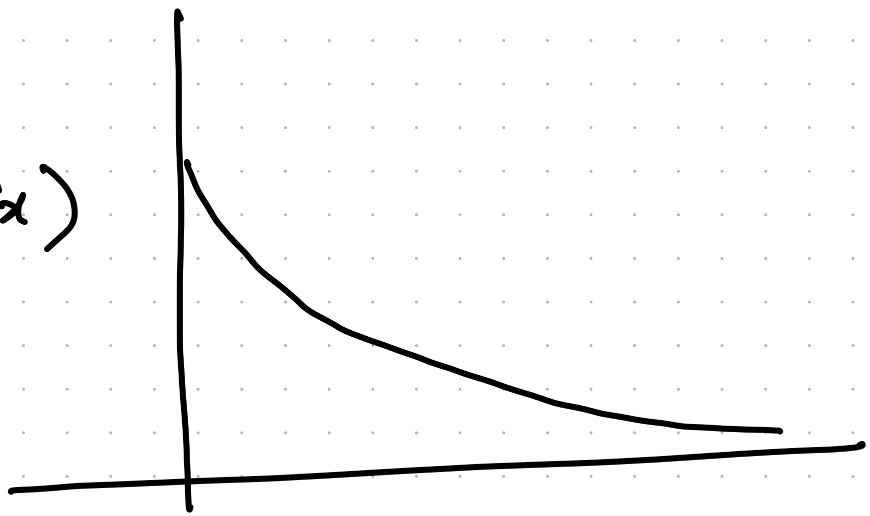

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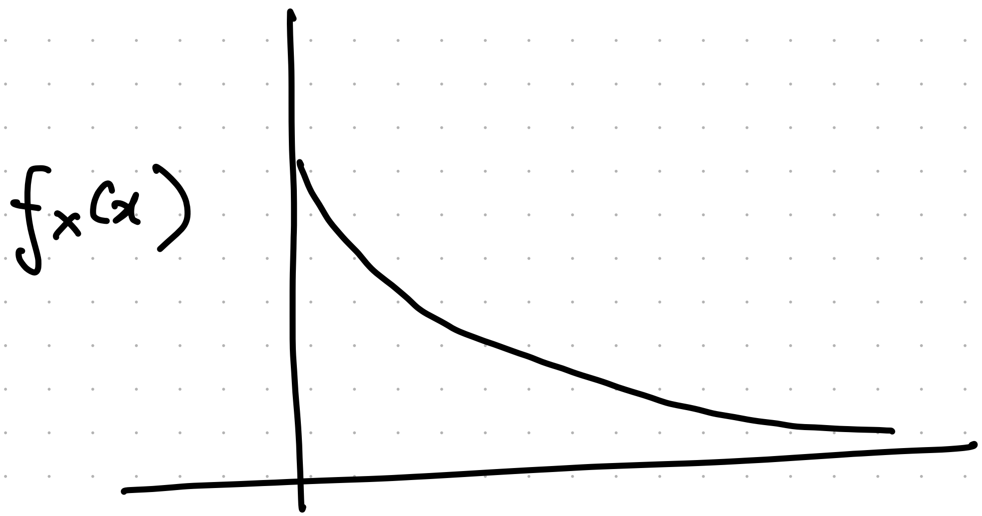

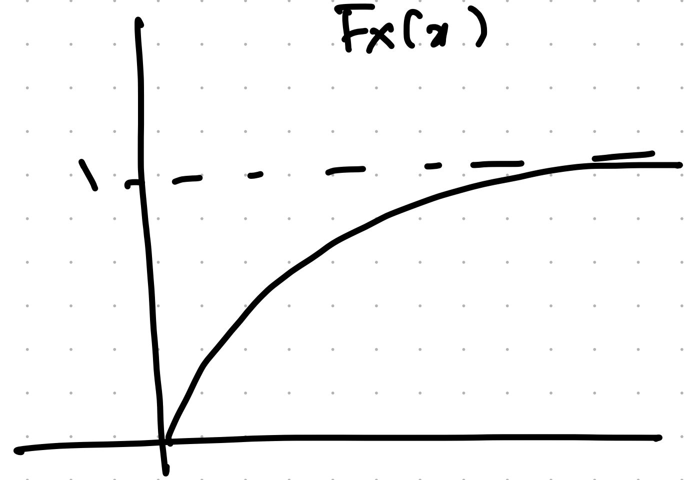

$$F_X(x) = \begin{cases} \displaystyle\int_{-\infty}^{x} \lambda e^{-\lambda t}\, dt = \int_{0}^{x} \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Q) For exp. r.v. with $f_X(x) = \lambda e^{-\lambda x}$

Find $P[1 \leq x \leq 3]$

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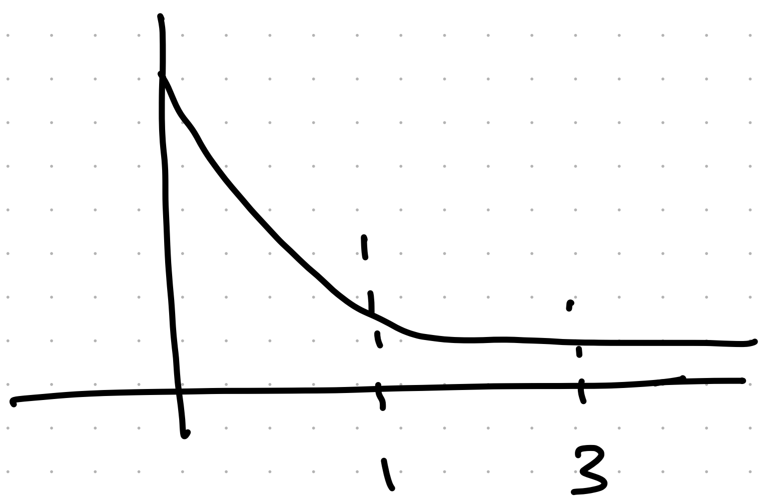

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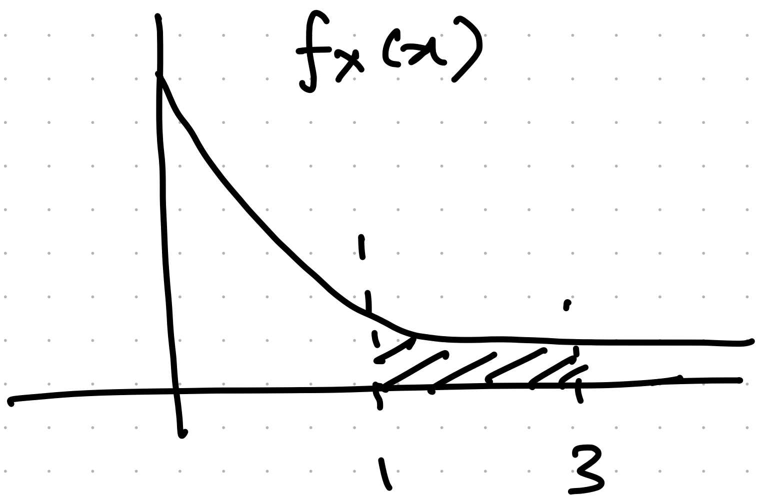

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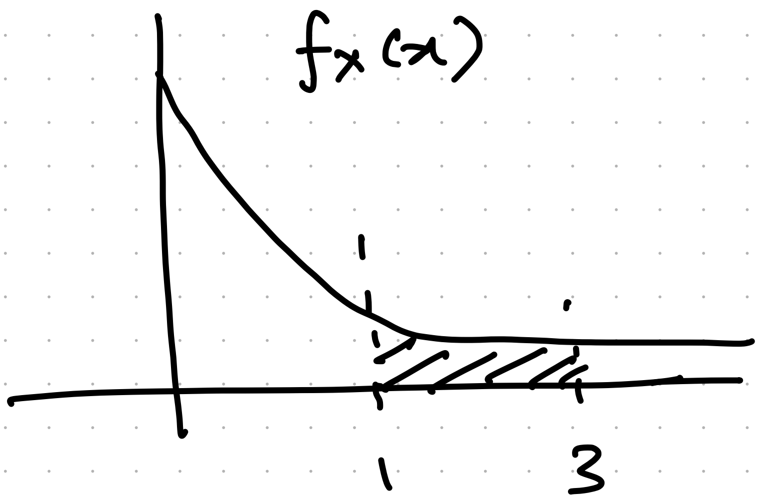

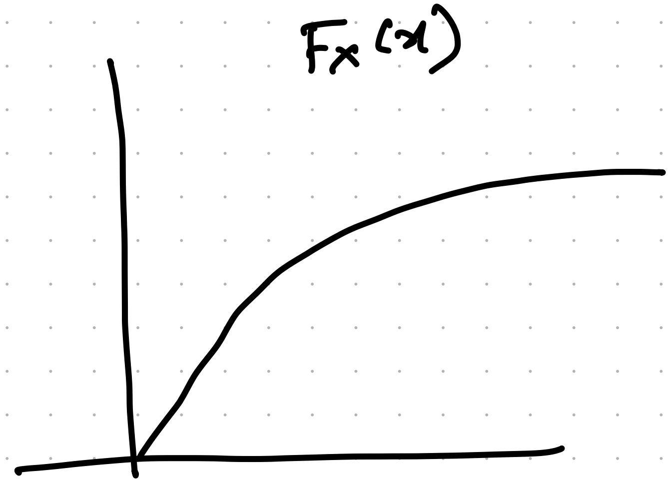

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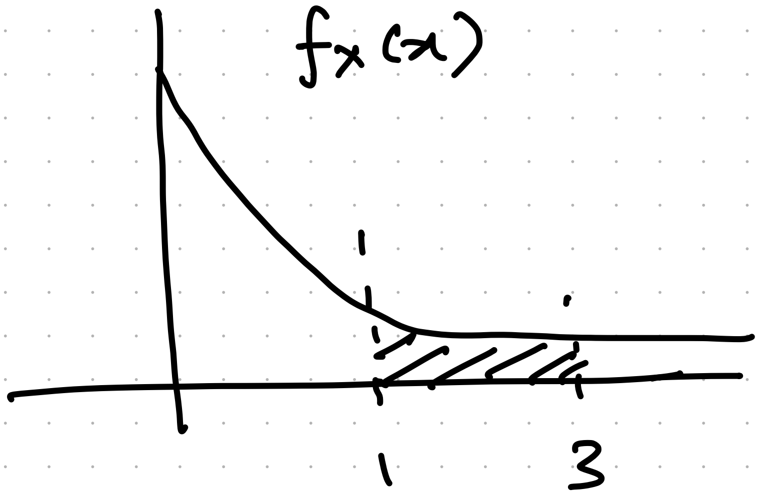

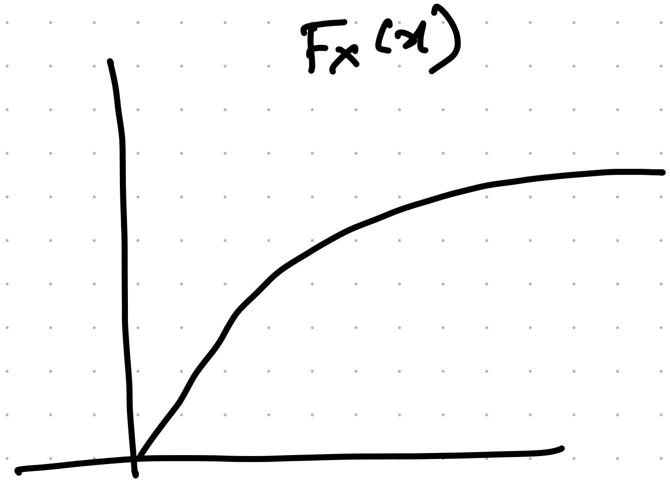

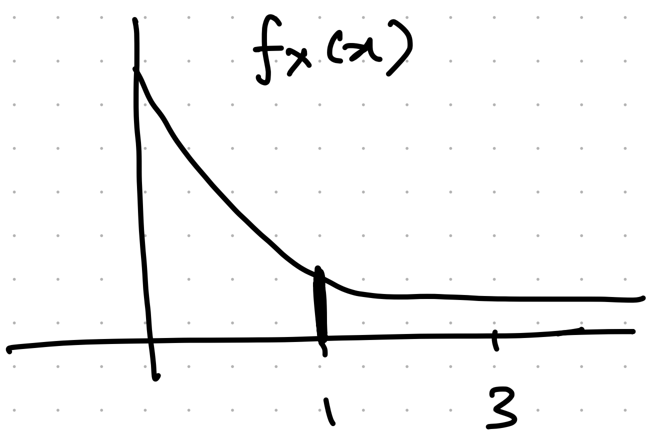

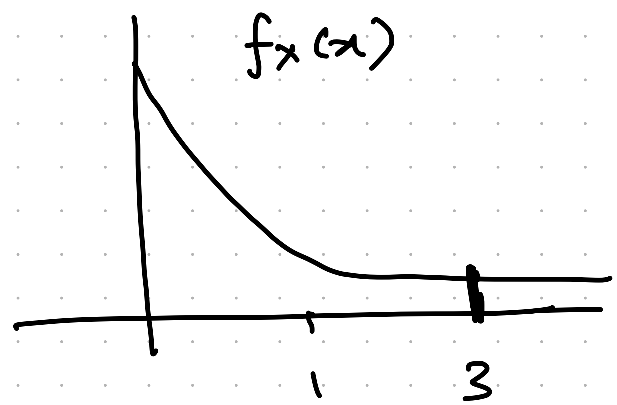

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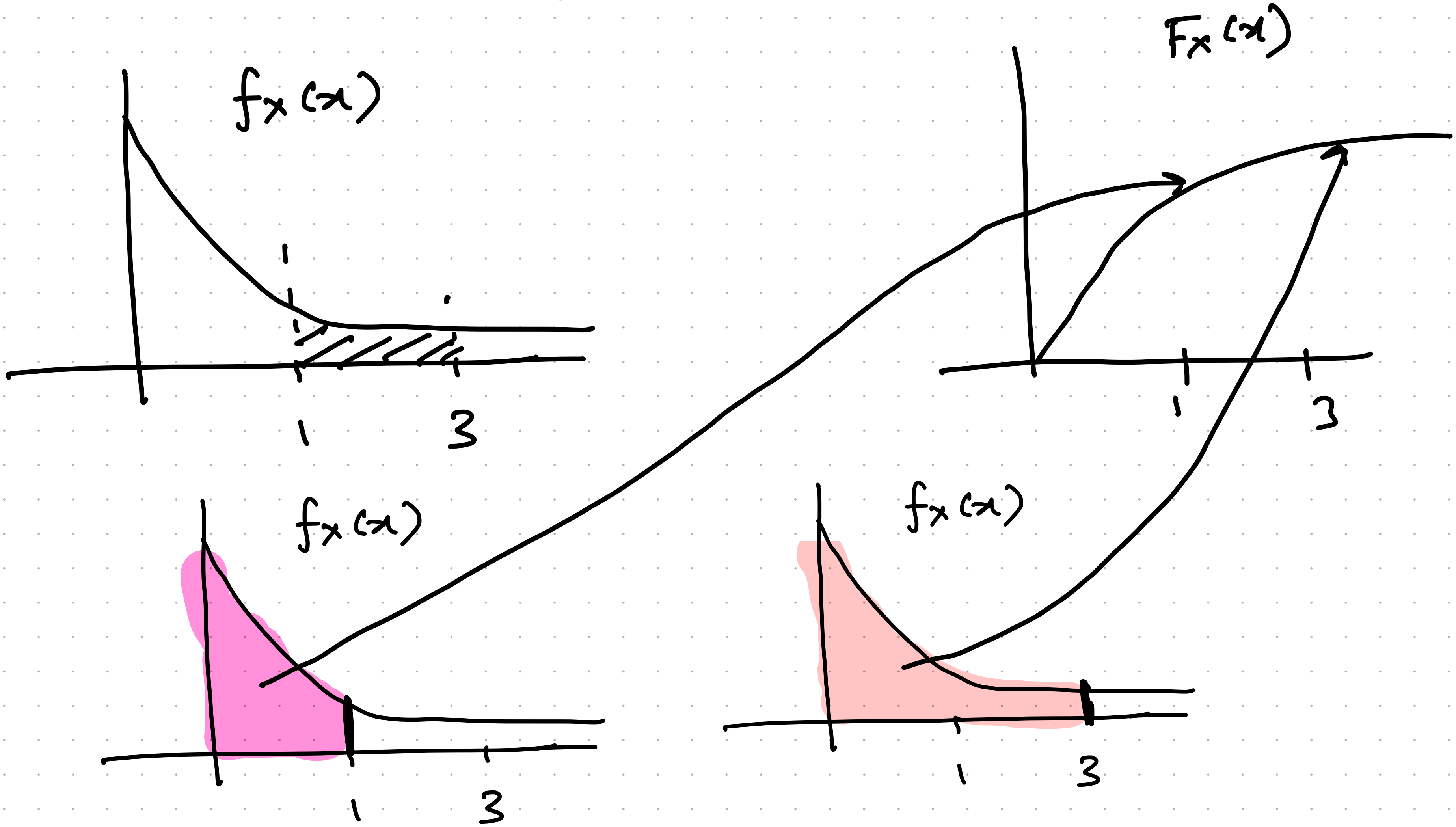

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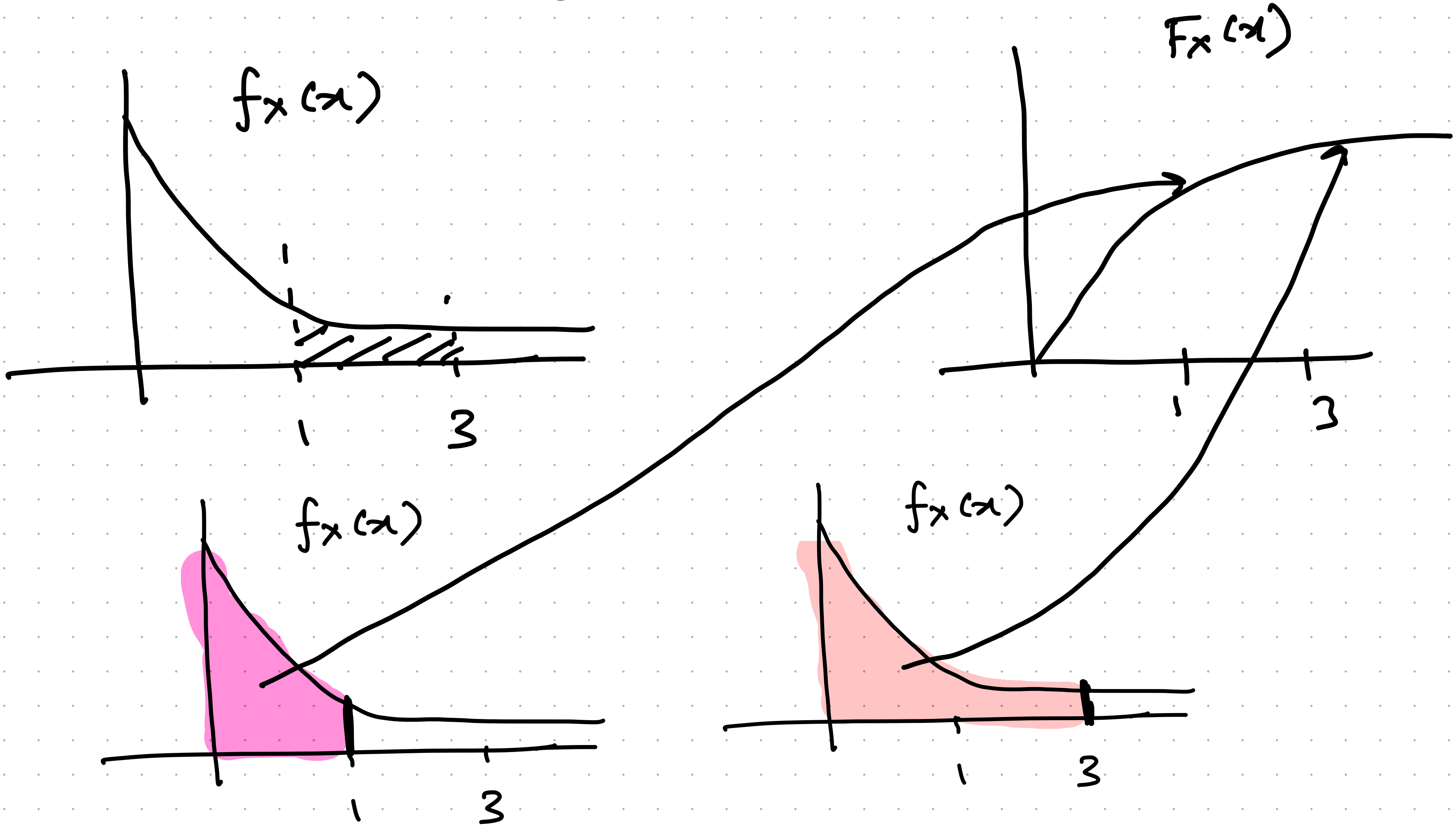

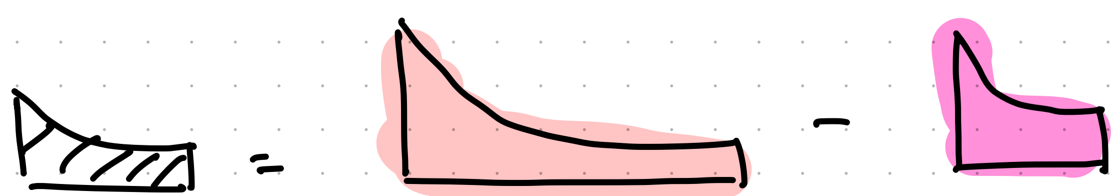

$= F_X(3) - F_X(1)$

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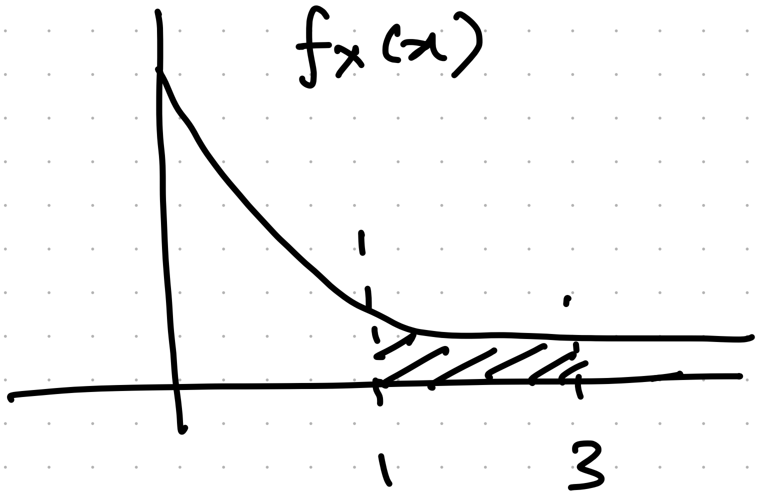


$f_X(x)$

$$P[1 \le x \le 3] = F_X(3) - F_X(1)$$

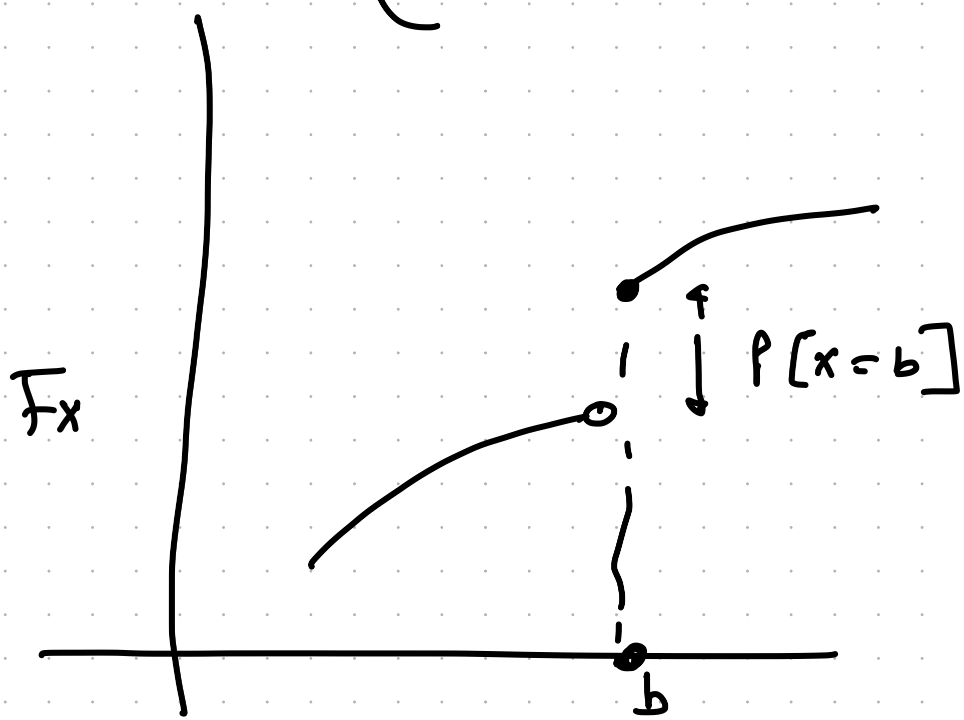
$$= 1 - e^{-3\lambda} - (1 - e^{-\lambda})$$

$$= e^{-\lambda} - e^{-3\lambda}$$

Revisiting jump property

For any r.v. X ; $P[X=b)$ is

$$\begin{cases} F_X(b) - F_X(b^-) & ; F_X \text{ discontinuous at } x=b \\ 0 & \text{o/w} \end{cases}$$



Q)

$$f_X(x) = \begin{cases} x \; ; & 0 \le x \le 1 \\ \frac{1}{2} \; ; & x = 3 \\ 0 \; ; & o/w \end{cases}$$

Find CDF

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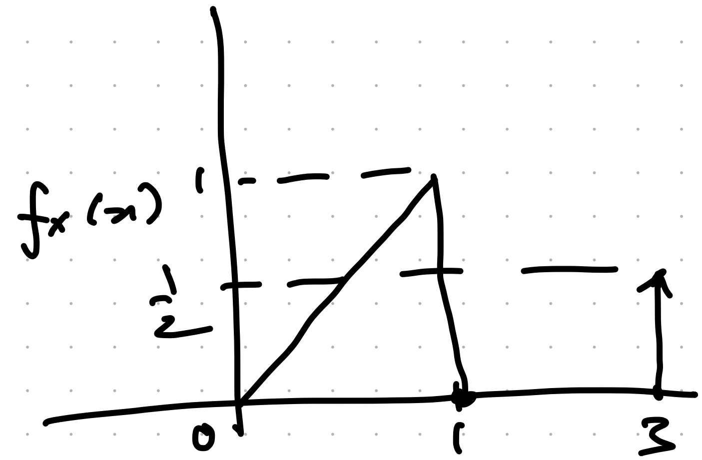

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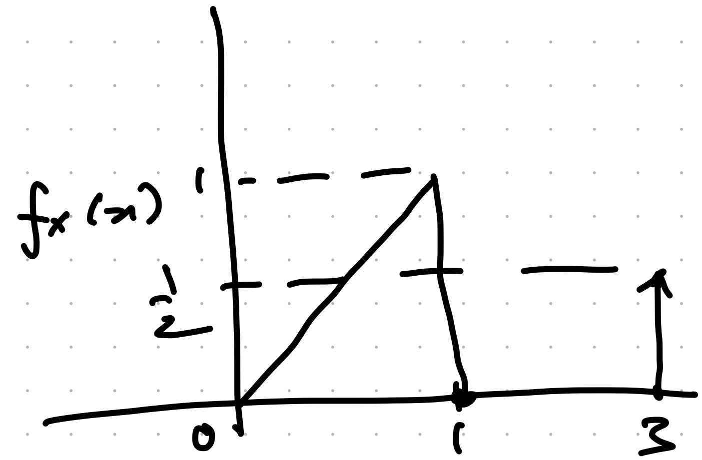

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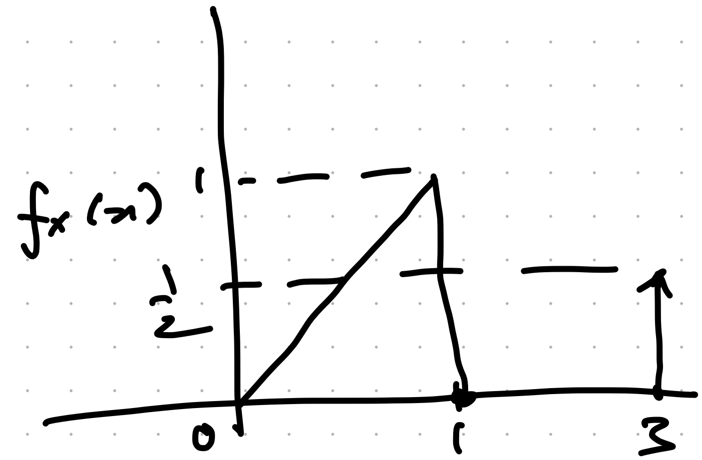

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b) $0 \le x \le 1$; $F_X(x) = \int_0^x t\,dt = \frac{x^2}{2}$

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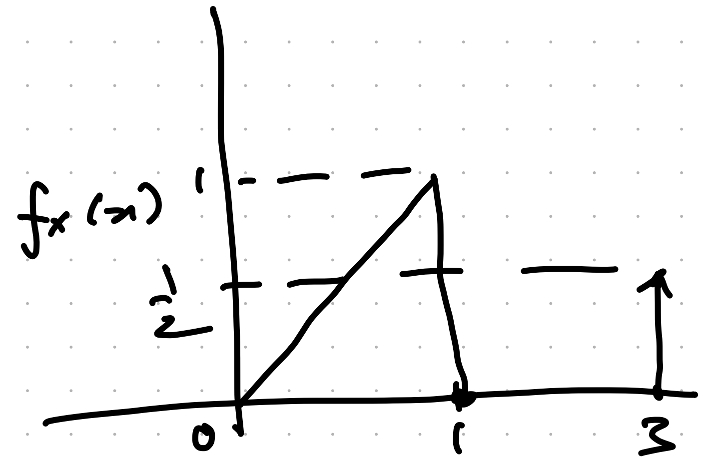

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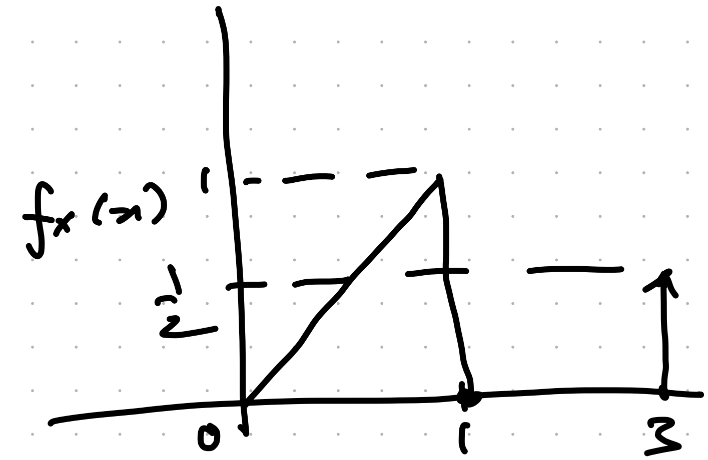
c) $1 < x < 3$; $F_X(x) = F_X(1) = \frac{1}{2}$

Q) $f_X(x) = \begin{cases} x ; & 0 \le x \le 1 \\ \frac{1}{2} ; & x = 3 \\ 0 ; & o/w \end{cases}$

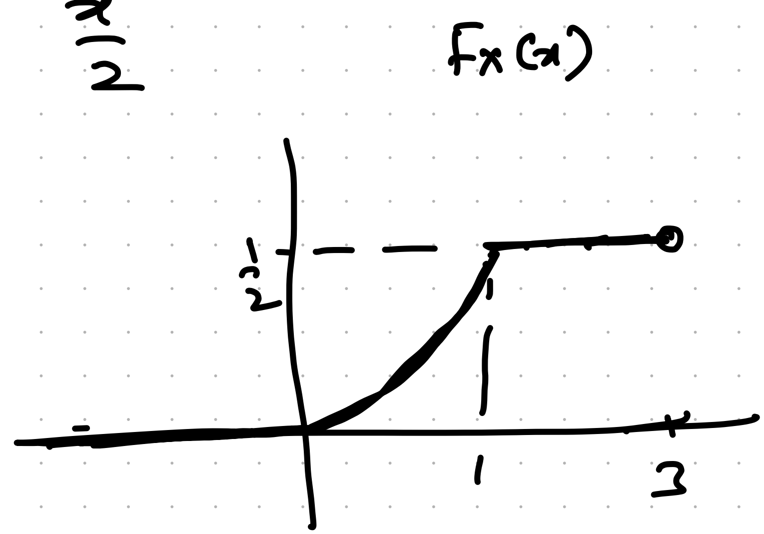


Find CDF

Hybrid r.v.

a) $x < 0$; $F_X(x) = 0$

b) $0 \le x \le 1$; $F_X(x) = \int_0^x t\,dt = \frac{x^2}{2}$



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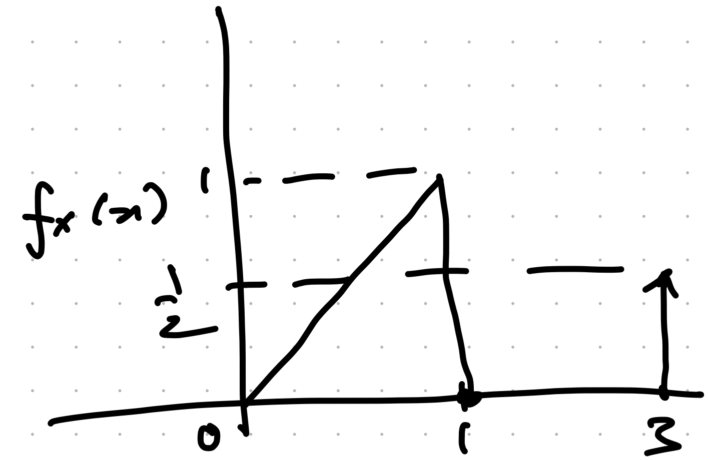

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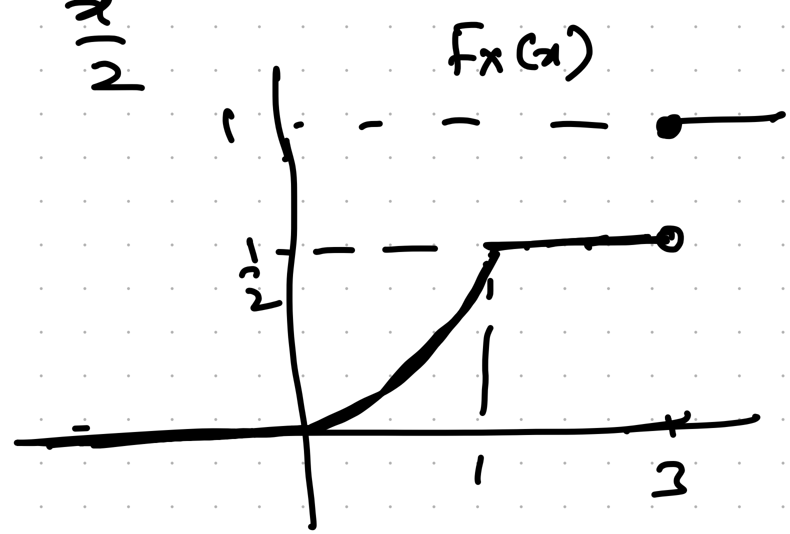

c) $1 < x < 3$; $F_X(x) = F_X(1) = \frac{1}{2}$

d) $x = 3$; $F_X(x) = F_X(3^-) + P(X=3)$

$= \frac{1}{2} + \frac{1}{2} = 1$

Deriving PDF from CDF

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} \int_{-\infty}^{x} f_X(x')\,dx'$$

if F_X is differentiable at x;

else

$$f_X(x) = P[X=x] = F_X(x) - F_X(x^-)$$

Q) $$F_X(x) = \begin{cases} 0 & ; x < 0 \\ 1 - \frac{1}{4} e^{-2x} & ; x \geq 0 \end{cases}$$

Find PDF

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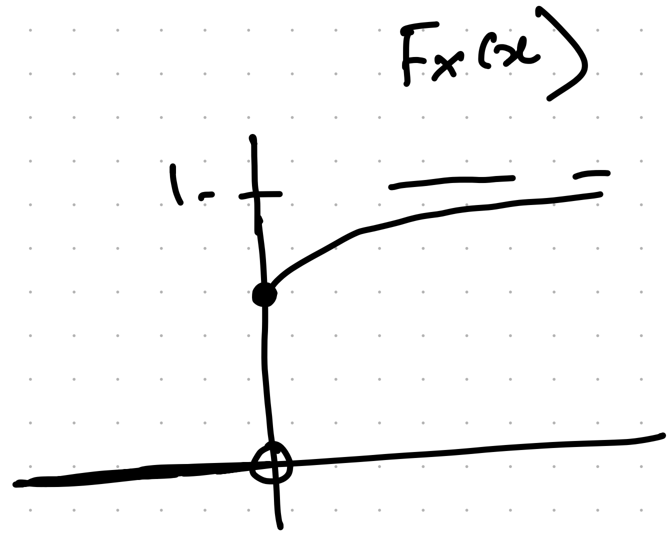

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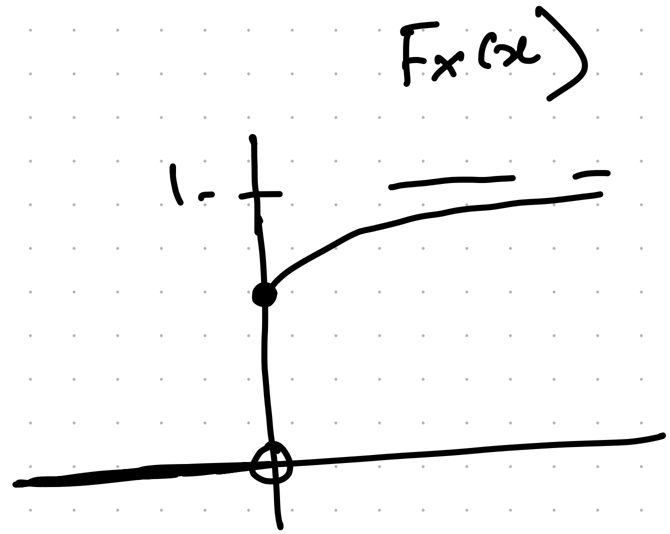

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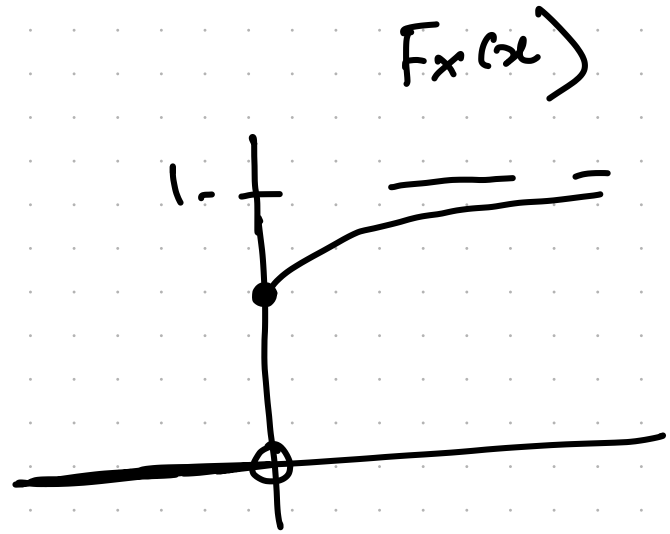

$x < 0$; $F_X(x) = 0$

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$$f_X(x) = \begin{cases} \frac{d}{dx} 0 & ; x < 0 \\ F_X(0) - F_X(0^-) & ; x = 0 \\ \frac{d}{dx}\left(1 - \frac{1}{4} e^{-2x}\right) & ; x > 0 \end{cases}$$

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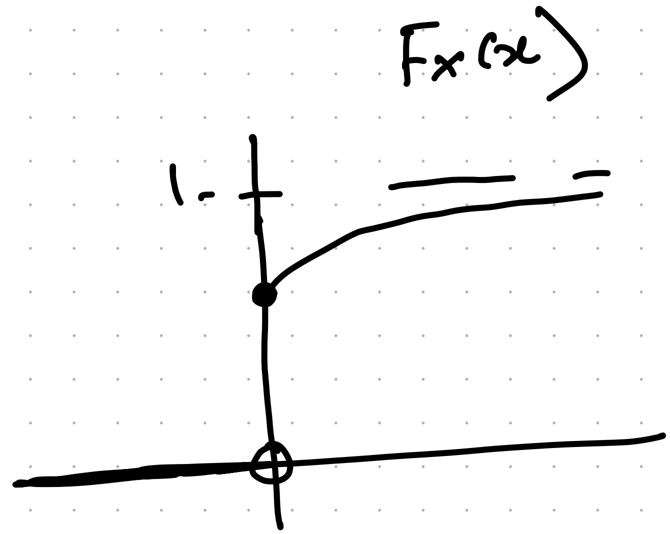

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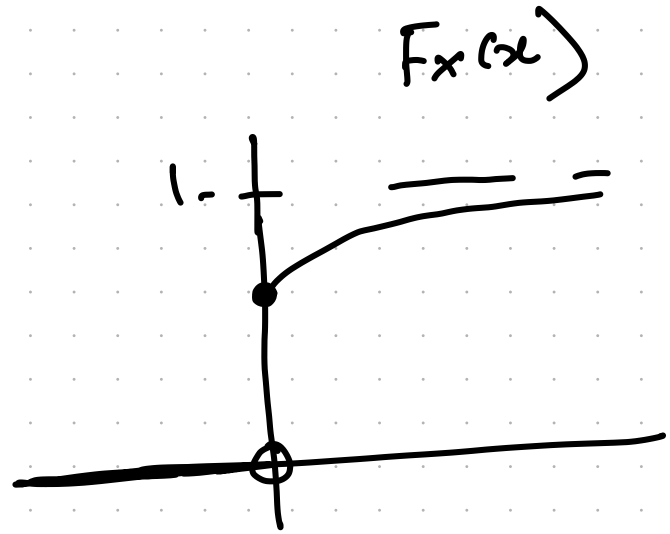

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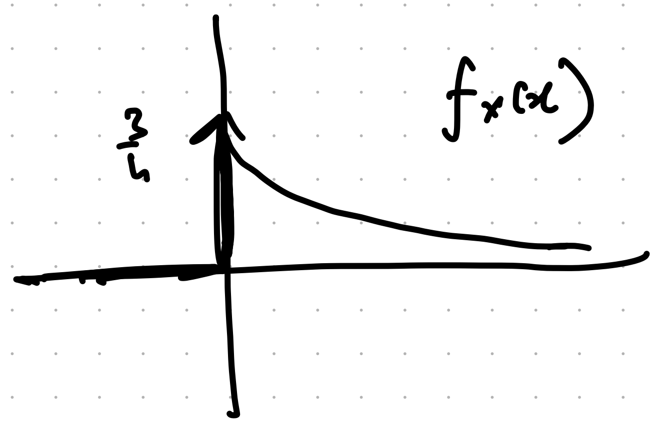

Inverse CDF Sampling

① Assume: we know how to sample from uniform dist $U(0,1)$

② Goal: Sample from r.v. X with PDF $f_X(x)$

CDF $F_X(x)$

Inverse CDF Sampling

Assume $F_X(x)$ to be:

- continuous
- strictly increasing

Inverse CDF Sampling

Assume $F_X(x)$ to be:

- continuous
- strictly increasing

Goal: learn strictly <u>monotone</u> transformation

$T : [0,1] \to R$

s.t.

$T(U) \overset{d}{=} X$

same dist. as X

__Aside__ (strictly monotone)

$T : [0,1] \to R$ is strictly monotone if:

1) Strictly increasing

$$x_1 < x_2 \Rightarrow T(x_1) < T(x_2)$$

2) strictly decreasing

$$x_1 < x_2 \Rightarrow T(x_1) > T(x_2)$$

Aside (Strictly monotone)

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$$x_1 < x_2 \Rightarrow T(x_1) > T(x_2)$$

Strict monotonicity ensures T^{-1} is defined.

Aside (Strictly monotone)

$T: [0,1] \to R$ is strictly monotone if:

$$T(u) = \ln u$$

$$u_1 < u_2 \Rightarrow \ln u_1 < \ln u_2$$

$$T^{-1}(x) = e^{x}$$

Strictly monotone

$$T(u) = \sin(\pi u)$$

Not 1-to-1

$\therefore$ Not monotonic $\Rightarrow$ T^{-1} not defined on all range

Inverse CDF Sampling

Assume $F_X(x)$ to be:

- continuous
- strictly increasing

Goal: learn strictly <u>monotone</u> transformation

$T: [0,1] \to \mathbb{R}$

s.t.

$T(U) \overset{d}{=} X$

same dist. as X

Inverse CDF Sampling

$T : [0,1] \rightarrow R$

s. t.

$T(U) \overset{d}{=} X$

$$F_X(x) = P[X \leq x] = P[T(U) \leq x]$$
$$= P[U \leq T^{-1}(x)]$$
$$= T^{-1}(x)$$

$\therefore \; F_X(x) = T^{-1}(x)$

or; $T(u) = F_X^{-1}(u)$; $u \in [0,1]$

Inverse CDF sampling procedure

1) Generate $u_1, \dots, u_n$ samples from U

2) Apply $x_i = F_X^{-1}(u_i)$ to get samples from X

Inverse CDF sampling procedure

$X \sim \text{Exp}(\lambda)$

Generate samples

Inverse CDF sampling procedure

$X \sim Exp(\lambda)$

Generate samples

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & ; x \geq 0 \\ 0 & ; o/w \end{cases}$$

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Consider $x \geq 0$;

$$F_X(x) = 1 - e^{-\lambda x} = y \text{ (say)}$$

Inverse CDF sampling procedure

$X \sim Exp(\lambda)$

Generate samples

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & ; x \geq 0 \\ 0 & ; o/w \end{cases}$$

Consider $x \geq 0$;

$$F_X(x) = 1 - e^{-\lambda x} = y \text{ (say)}$$

$$1 - y = e^{-\lambda x}$$

$$\log(1-y) = -\lambda x \qquad \text{or } x = -\frac{1}{\lambda}\log(1-y)$$

$$\therefore F_X^{-1}(u) = -\frac{1}{\lambda}\log(1-u)$$