# ES114: Probability, Statistics, and Data Visualization

## Conditional Distributions

$1^{st}$ April 2025

# Recap

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  - Joint Expectations
  - Covariance and Correlation
- Conditional Distribution

# Recap

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  - Covariance and Correlation
- Conditional Distribution
- Conditional Expectation

# References

- https://probability4datascience.com/slides/Slide_5_04.pdf
- https://probability4datascience.com/slides/Slide_5_05.pdf
- Other references
  - https://online.stat.psu.edu/stat414/lesson/19/19.1
  - https://stats.libretexts.org/Bookshelves/Probability_Theory/Introductory_Probability_(Grinstead_and_Snell)/04%3A_Conditional_Probability/4.02%3A_Continuous_Conditional_Probability

# Safety measure vs Injury

$X \quad , Y \leftarrow$

| $f(x,y)$ | Safety Measure ($Y$) | | |
|---|---|---|---|
| Injury ($X$) | None (0) | Belt Only (1) | Belt and Harness (2) |
| None (0) | $\boxed{0.065}$ + | 0.075 + | 0.060 |
| Minor (1) | 0.175 | 0.160 | 0.115 |
| Major (2) | 0.135 | 0.100 | 0.065 |
| Death (3) | 0.025 | 0.015 | 0.010 |

$f(Y=0)$

Table: Joint Probability Table

$f.(X=0)$

$$\sum_x \sum_y f(x,y) = 1$$

# Safety measure vs Injury

| $f(x,y)$ | Safety Measure ($Y$) | | | $f_X(x)$ |
|---|---|---|---|---|
| Injury ($X$) | None (0) | Belt Only (1) | Belt and Harness (2) | |
| None (0) | 0.065 | 0.075 | 0.060 | 0.200 |
| Minor (1) | 0.175 | 0.160 | 0.115 | 0.450 |
| Major (2) | 0.135 | 0.100 | 0.065 | 0.300 |
| Death (3) | 0.025 | 0.015 | 0.010 | 0.050 |
| $f_Y(y)$ | 0.400 | 0.350 | 0.250 | 1.000 |

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What is the probability that a randomly selected person was wearing only a seat belt and had only a minor injury?

# Safety measure vs Injury

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What is the probability that a randomly selected person was wearing only a seat belt and had only a minor injury? $0.16$

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# Safety measure vs Injury

| $f(x,y)$ | Safety Measure ($Y$) | | | $f_X(x)$ |
|---|---|---|---|---|
| Injury ($X$) | None (0) | Belt Only (1) | Belt and Harness (2) | |
| None (0) | 0.065/0.4 | 0.075 | 0.060 | 0.200 |
| Minor (1) | 0.175 | 0.160 | 0.115 | 0.450 |
| Major (2) | 0.135/ | 0.100 | 0.065 | 0.300 |
| Death (3) | 0.025/0.4 | 0.015 | 0.010 | 0.050 |
| $f_Y(y)$ | 0.400 | 0.350 | 0.250 | 1.000 |

If a randomly selected person takes no safety measure, what is the probability of death?

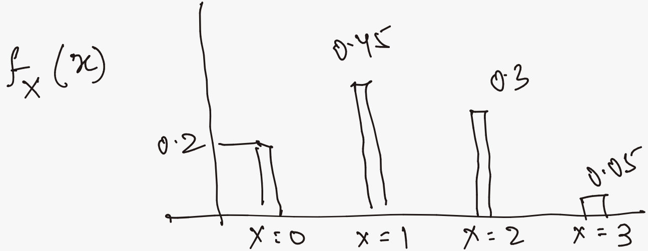
$$Pr(X=3 \mid Y=0) = \frac{Pr(X=3, Y=0)}{Pr(Y=0)} = \frac{0.025}{0.4}$$

$$\Pr(X=0 \mid Y=0) = \frac{0.065}{0.4} = 0.1625$$

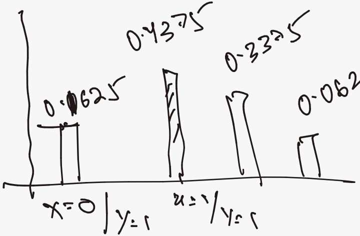
$$\Pr(X=1 \mid Y=0) = \frac{0.175}{0.4}$$

$$\Pr(X=2 \mid Y=0) = \frac{\Pr(X=2, Y=0)}{\Pr(Y=0)} = \frac{0.135}{0.4}$$

$$\Pr(X=3 \mid Y=0) = \frac{0.025}{0.4}$$




$f_X(x \mid Y=0)$



# Safety measure vs Injury

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

| $f(x,y)$ | Safety Measure ($Y$) | | | $f_X(x)$ |
|---|---|---|---|---|
| Injury ($X$) | None (0) | Belt Only (1) | Belt and Harness (2) | |
| None (0) | 0.065 /0.2 | 0.075 / 0.2 | 0.060 /0.2 | 0.200 |
| Minor (1) | 0.175 /0.45 | 0.160 / 0.45 | 0.115 | 0.450 |
| Major (2) | 0.135 | 0.100 | 0.065 | 0.300 |
| Death (3) | 0.025 | 0.015 | 0.010 | 0.050 |
| $f_Y(y)$ | 0.400 | 0.350 | 0.250 | 1.000 |

# Safety measure vs Injury

$f_{X|Y}(x|y)$

| $f(x,y)$ | Safety Measure ($Y$) | | | $f_X(x)$ |
|---|---|---|---|---|
| Injury ($X$) | None (0) | Belt Only (1) | Belt and Harness (2) | |
| None (0) | 0.065/0.4 | 0.075/0.350 | 0.060 | 0.200 |
| Minor (1) | 0.175/0.4 | 0.160/0.350 | 0.115 | 0.450 |
| Major (2) | 0.135/0.4 | 0.100/0.350 | 0.065 | 0.300 |
| Death (3) | 0.025/0.4 | 0.015/0.350 | 0.010 | 0.050 |
| $f_Y(y)$ | 0.400 | 0.350 | 0.250 | 1.000 |

If a randomly selected person takes no safety measure, what is the probability of $X$ ?

# Conditional Distribution

The distribution of subpopulation having some characteristic fixed.

$$f_{X|Y}(x \mid y = 1)$$

$$f_{X|Y}(x \mid y = 2)$$

# Conditional Distribution

- The conditional PMF

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

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- The conditional PMF

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

- $0 \leq f_{X|Y}(x|y) \leq 1$

# Conditional Distribution

- The conditional PMF

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

- $0 \leq f_{X|Y}(x|y) \leq 1$
- $\sum_x f_{X|Y}(X = x|Y = y) = ?$

# Conditional Distribution

- The conditional PMF

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

- $0 \leq f_{X|Y}(x|y) \leq 1$
- $\sum_x f_{X|Y}(X = x|Y = y) =?$ $\underline{1}$
- $\sum_y f_{X|Y}(X = x|Y = y) =?$

# Conditional Distribution

| $f_{X\|Y}(x\|y)$ | Safety Measure ($Y$) | | | $\sum_y f_{X\|Y}(x\|y)$ |
|---|---|---|---|---|
| Injury ($X$) | None (0) | Belt Only (1) | Belt and Harness (2) | |
| None (0) | 0.1625 | 0.2143 | 0.2400 | 0.6168 |
| Minor (1) | 0.4375 | 0.4571 | 0.4600 | 1.3546 |
| Major (2) | 0.3375 | 0.2857 | 0.2600 | 0.8832 |
| Death (3) | 0.0625 | 0.0429 | 0.0400 | 0.1454 |
| $\sum_x f_{X\|Y}(x\|y)$ | 1 | 1 | 1 | |

# Conditional Distribution

| $f_{Y\|X}(y\|x)$ | Safety Measure ($Y$) | | | $\sum_y f_{X\|Y}(x\|y)$ |
|---|---|---|---|---|
| Injury ($X$) | None (0) | Belt Only (1) | Belt and Harness (2) | |
| None (0) | 0.325 | 0.375 | 0.300 | 1 |
| Minor (1) | 0.3889 | 0.3556 | 0.2556 | 1 |
| Major (2) | 0.450 | 0.3333 | 0.2167 | 1 |
| Death (3) | 0.500 | 0.300 | 0.200 | 1 |
| $\sum_x f_{X\|Y}(x\|y)$ | 1.164 | 1.364 | 0.972 | |

# Conditional Distribution

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| $\sum_x f_{X\|Y}(x\|y)$ | 1.164 | 1.364 | 0.972 | |

$$\boxed{f_{X|Y}(x|y) \neq f_{Y|X}(y|x)}$$

# Joint to ...

| $f(x,y)$ | Safety Measure ($Y$) | | |
|---|---|---|---|
| Injury ($X$) | None (0) | Belt Only (1) | Belt and Harness (2) |
| None (0) | 0.065 | 0.075 | 0.060 |
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# Joint to ... $y>0 \mid x : +1$

$x + N > 0$

$N > -x$

$N > (-1, \infty)$

$x : +1$

$y > 0$

| $f(x,y)$ | Safety Measure ($Y$) | | |
|---|---|---|---|
| Injury ($X$) | None (0) | Belt Only (1) | Belt and Harness (2) |
| None (0) | 0.065 | 0.075 | 0.060 |
| Minor (1) | 0.175 | 0.160 | 0.115 |
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| Death (3) | 0.025 | 0.015 | 0.010 |

Given the above table we can get $f_X(x)$, $f_Y(y)$, $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$

# … to Joint

| $f(x,y)$ | Safety Measure ($Y$) | | | $f_X(x)$ |
|---|---|---|---|---|
| Injury ($X$) | None (0) | Belt Only (1) | Belt and Harness (2) | |
| None (0) | $a_1$ | $a_2$ | $a_3$ | 0.200 |
| Minor (1) | $a_4$ | $a_5$ | $a_6$ | 0.450 |
| Major (2) | $a_7$ | $a_8$ | $a_9$ | 0.300 |
| Death (3) | $a_{10}$ | $a_{11}$ | $a_{12}$ | 0.050 |
| $f_Y(y)$ | 0.400 | 0.350 | 0.250 | 1.000 |

# Varids 12

$$a_1 + a_4 + a_7 + a_{10} = 0.4$$

$$\vdots$$

# Eqn 7

# … to Joint

| $f(x,y)$ | Safety Measure ($Y$) | | | $f_X(x)$ |
|---|---|---|---|---|
| Injury ($X$) | None (0) | Belt Only (1) | Belt and Harness (2) | |
| None (0) | | | | 0.200 |
| Minor (1) | | | | 0.450 |
| Major (2) | | | | 0.300 |
| Death (3) | | | | 0.050 |
| $f_Y(y)$ | 0.400 | 0.350 | 0.250 | 1.000 |

- Can you get $f(x,y)$? Not unique

## … to Joint

$f(x,y) = f_x(x)\, f_y(y) \longrightarrow$ Independent

| $f(x,y)$ | Safety Measure ($Y$) | | | $f_X(x)$ |
|---|---|---|---|---|
| Injury ($X$) | None (0) | Belt Only (1) | Belt and Harness (2) | |
| None (0) | $0.4 \times 0.2$ | | | 0.200 |
| Minor (1) | | | | 0.450 |
| Major (2) | | | | 0.300 |
| Death (3) | | | | 0.050 |
| $f_Y(y)$ | 0.400 | 0.350 | 0.250 | 1.000 |

- Can you get $f(x,y)$?
- What if $X$ and $Y$ are independent?

# ... to Joint

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| None (0) | | | | 0.200 |
| Minor (1) | | | | 0.450 |
| Major (2) | | | | 0.300 |
| Death (3) | | | | 0.050 |
| $f_Y(y)$ | 0.400 | 0.350 | 0.250 | 1.000 |

- Can you get $f(x,y)$?
- What if $X$ and $Y$ are independent?
- What if you know $f_{X|Y}(x|y)$ ?

$$f(x,y) = \underline{f_{X|Y}(x|y)}\ \underline{f_Y(y)}$$

# Conditional Distribution

Consider two random variables $X$ and $Y$ defined as follows:

$$Y = \begin{cases} 10^2, & \text{with probability } \frac{5}{6} \\ 10^4, & \text{with probability } \frac{1}{6} \end{cases}$$

$$X = \begin{cases} 10^{-4}Y, & \text{with probability } \frac{1}{2} \\ 10^{-3}Y, & \text{with probability } \frac{1}{3} \\ 10^{-2}Y, & \text{with probability } \frac{1}{6} \end{cases}$$

# Conditional Distribution

$$f(x \mid y = 10^4) = \begin{cases} 1 & 1/2 \\ 10 & 1/3 \\ 10^2 & 1/6 \end{cases}$$

Consider two random variables $X$ and $Y$ defined as follows:

| y \ x | $10^{-2}$ | $10^{-1}$ | $1$ | $10$ | $10^2$ |
|---|---|---|---|---|---|
| $10^4$ | | | | | |
| $10^2$ | | | | | |

$$Y = \begin{cases} 10^2, & \text{with probability } \frac{5}{6} \\ 10^4, & \text{with probability } \frac{1}{6} \end{cases}$$

$$X = \begin{cases} 10^{-4}Y, & \text{with probability } \frac{1}{2} \\ 10^{-3}Y, & \text{with probability } \frac{1}{3} \\ 10^{-2}Y, & \text{with probability } \frac{1}{6} \end{cases}$$

Find $f_{X,Y}(x, y)$

$$f(x \mid y = 10^2) = \begin{cases} 10^{-2} & 1/2 \\ 10^{-1} & 1/3 \\ 1 & 1/6 \end{cases}$$

$Pr(x > 1 \mid y = 10^4)$

| $f_Y(y)$ | $f(x \mid y)$ $y \backslash x$ | 0.01 | 0.1 | 1 | 10 | 100 |
|---|---|---|---|---|---|---|
| $\frac{1}{6}$ | $y = 10^4$ | 0 | 0 | 1/2 | 1/3 | 1/6 |
| 5/6 | $10^2$ | 1/2 | 1/3 | 1/6 | 0 | 0 |

$= 1$

| $f(x, y)$ $y \backslash x$ | 0.01 | 0.1 | 1 | 10 | 100 |
|---|---|---|---|---|---|
| $y$: $10^4$ | 0 | 0 | $\frac{1}{12}$ | $\frac{1}{18}$ | $\frac{1}{36}$ |
| $10^2$ | $\frac{5}{12}$ | $\frac{5}{18}$ | $\frac{5}{36}$ | 0 | 0 |

$= 1$

# Conditional Distribution (PMF)

Consider two random variables $X$ and $Y$ defined as follows:

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Find $\Pr(X > 1 | Y = 10^4)$ $= \sum_{x>1} f_{X|Y}(x \mid Y = 10^4)$

$= f_{X|Y}(x = 10 \mid Y = 10^4) + f_{X|Y}(x = 100 \mid Y = 10^4)$

# Conditional Distribution (PMF)

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$$Y = \begin{cases} 10^2, & \text{with probability } \frac{5}{6} \\ 10^4, & \text{with probability } \frac{1}{6} \end{cases}$$

$$X = \begin{cases} 10^{-4}Y, & \text{with probability } \frac{1}{2} \\ 10^{-3}Y, & \text{with probability } \frac{1}{3} \\ 10^{-2}Y, & \text{with probability } \frac{1}{6} \end{cases}$$

Find $\Pr(X > 1)$

$$= \sum_{y \in \{10^2, 10^4\}} P(X > 1, Y = y)$$

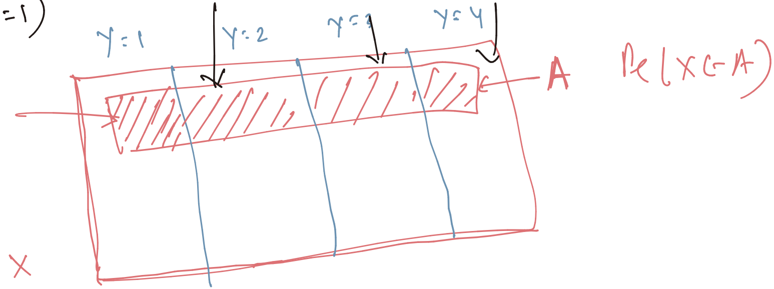
# Conditional Distribution (Characteristics)

- $\Pr(X \in A | Y = y) = \sum_{x \in A} f_{X|Y}(x|y)$

# Conditional Distribution (Characteristics)

- $\Pr(X \in A | Y = y) = \sum_{x \in A} f_{X|Y}(x|y)$
- Total Probability $\Pr(X \in A)$ ?

$Pr(X \in A | Y=1) Pr(Y=1)$



# Conditional Distribution (Characteristics)

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# Conditional Distribution (Characteristics)

- $\Pr(X \in A | Y = y) = \sum_{x \in A} f_{X|Y}(x|y)$ ← Eq 1
- Total Probability $\Pr(X \in A)$ ?

$$\Pr(X \in A) = \sum_{Y=y} \Pr(X \in A, Y = y)$$
$$= \sum_{Y=y} \Pr(X \in A | Y = y) f_Y(y) \qquad \text{Using Eq 1}$$
$$= \boxed{\sum_{x \in A} \sum_{Y=y} f_{X|Y}(x|y) f_Y(y)}$$

# Example

| $f(X,Y)$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | $1/20$ | $1/20$ | $1/20$ | 0 |
| 2 | $1/20$ | $2/20$ | $3/20$ | $1/20$ |
| 3 | $1/20$ | $2/20$ | $3/20$ | $1/20$ |
| 4 | 0 | $1/20$ | $1/20$ | $1/20$ |

# Example

| $f(X,Y)$ | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|
| 1 | 1/20 | 1/20 | 1/20 | 0 | 3/20 |
| 2 | 1/20 | 2/20 | 3/20 | 1/20 | 7/20 |
| 3 | 1/20 | 2/20 | 3/20 | 1/20 | 7/20 |
| 4 | 0 | 1/20 | 1/20 | 1/20 | 3/20 |

Find $\Pr[X > 2 | Y = 1]$ and $\Pr[X > 2]$

$\sum_{y=} \Pr[X>2 | Y=y] \Pr(Y=y)$ $\quad \sum_{y \in \{1,2,3,4\}} \Pr[X>2, y]$

# Continuous Conditional Distribution

$X$ is a continuous random variable

- $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$

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- $\Pr[X \in A|Y = y] = \int_{x \in A} f_{X|Y}(x|y)dx$

# Continuous Conditional Distribution

$X$ is a continuous random variable

- $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$
- $\Pr[X \in A | Y = y] = \int_{x \in A} f_{X|Y}(x|y) dx$
- $\Pr[X \in A] = ?$

$$\int_y \Pr[X \in A | Y = y] f_Y(y) dy$$

$$= \boxed{\int_y \int_{x \in A} f_{X|Y}(x|y) dx \, f_Y(y) dy}$$

# Continuous Conditional Distribution

Let $X$ be a random bit such that

$$X = \begin{cases} +1, & \text{with probability } \frac{1}{2} \\ -1, & \text{with probability } \frac{1}{2} \end{cases}$$

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Suppose that $X$ is transmitted over a noisy channel so that the observed signal is

$$Y = X + N,$$

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Suppose that $X$ is transmitted over a noisy channel so that the observed signal is

$$Y = X + N,$$

where $N \sim \mathcal{N}(0,1)$ is Gaussian noise, independent of the signal $X$.

# Continuous Conditional Distribution

Let $X$ be a random bit such that

$$X = \begin{cases} +1, & \text{with probability } \frac{1}{2} \\ -1, & \text{with probability } \frac{1}{2} \end{cases}$$

Suppose that $X$ is transmitted over a noisy channel so that the observed signal is

$$Y = X + N,$$

where $N \sim \mathcal{N}(0,1)$ is Gaussian noise, independent of the signal $X$. What is $\Pr(X = +1 | Y > 0)$?

$$\Pr[Y > 0 \mid X = +1] = \int_{-1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy$$

$$X + N > 0$$

$$N > -X$$

$$\Pr[y > 0 \mid x = +1] = \int_{-1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy$$

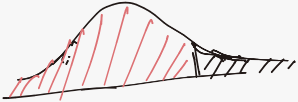
$\Phi$ - CDF of Gaussian

$$= 1 - \int_{-\infty}^{-1} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy$$

$$= 1 - \Phi(-1)$$

$$\Pr[y > 0 \mid x = -1] = 1 - \Phi(1)$$

$$\Pr[x = +1 \mid y > 0] = \frac{P(x = +1, y > 0)}{P(y > 0)}$$



$\Phi(1) + \Phi(-1) = 1$

$$= \frac{P(y>0 \mid x=+1)\,P(x=+1)}{P(y>0)}$$

$$= \frac{\left(1-\Phi(-1)\right)1/2}{\left(1-\Phi(-1)\right)1/2 + \left(1-\Phi(1)\right)1/2}$$

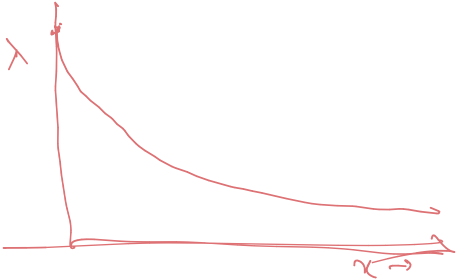
$$= \frac{\left(1-\Phi(-1)\right)1/2}{1-\frac{1}{2}\underbrace{\left(\Phi(-1)+\Phi(1)\right)}_{}}$$

$$= \boxed{\left(1-\Phi(-1)\right)}$$

# Continuous Conditional Distribution

Observing a lump of plutonium-239: Our experiment consists of waiting for an emission, then starting a clock, and recording the length of time $X$ that passes until the next emission. Experience has shown that $X$ has an exponential density with some parameter $\lambda$, which depends upon the size of the lump.

$$\lambda e^{-\lambda x}$$



# Continuous Conditional Distribution

$$\Pr[X>a] = \int_a^\infty \lambda e^{-\lambda x} = e^{-\lambda a}$$

$$\Pr[X>s]$$

Observing a lump of plutonium-239: Our experiment consists of waiting for an emission, then starting a clock, and recording the length of time $X$ that passes until the next emission. Experience has shown that $X$ has an exponential density with some parameter $\lambda$, which depends upon the size of the lump.

**Given that you have waited for $r$ seconds with no emission, what is the probability that there is no emission in further $s$ seconds?**

$$\boxed{s, r > 0} \rightarrow \Pr[X > s+r \mid X > r] = \frac{\Pr[X > s+r,\ X > r]}{\Pr[X > r]}$$

$$= e^{-\lambda(r+s)} / e^{-\lambda r} = \boxed{e^{-\lambda s}} = \Pr[X > s+r] / \Pr[X > r]$$

# Conditional Probability

When $X$ is independent of $Y$

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{f_X(x)f_Y(y)}{f_Y(y)} \\ &= f_X(x) \end{aligned}$$

# Conditional Expectation

- Discrete r.v.

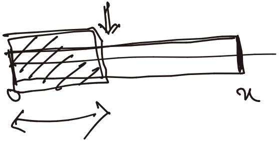
$$\mathbb{E}[X \mid Y = y] = \sum_x x f_{X|Y}(x \mid y)$$

- Continuous r.v.

$$\mathbb{E}[X \mid Y = y] = \int_x x f_{X|Y}(x \mid y)$$

# Example

$E_X[E[Y|X]] = E[Y]$




$f(y|X=x)$

$\int_0^x c\, dx = 1$

$\boxed{c = \frac{1}{x}}$

A stick of length one is broken at a random point, uniformly distributed over the stick. The remaining piece is broken once more.

- Find the expected length of the piece that now remains given the length of first piece

$\underline{X}$ : length of the first piece $(0,1)$

Unif$(0,1)$

Unif$(0,x)$

$Y$ : length of the second piece $(0,x)$

$f_{Y|X}(y|x) = \frac{1}{x}$

$E[Y|X=x] = \int_0^x y\, f(y|x)\, dy$

$= \int_0^x y \frac{1}{x}\, dy$

$$\int_0^x y\, f(y|x)\, dy$$

$$E[Y] = E_X\left[E[Y|X]\right]$$

$$\int_0^x y \frac{1}{x}\, dy$$

$$= \frac{1}{x}\left[\frac{y^2}{2}\right]_0^x = \frac{x}{2}$$

$$E[Y \mid X = x] = \frac{x}{2} \qquad E[Y]$$

$$E[y] = \underset{x}{E}\left[ E[y \mid x] \right] = E_x\left[ \frac{x}{2} \right]$$

$$E[y \mid x] = \frac{x}{2}$$

$$= \int_0^1 \frac{x}{2}\, dx$$

$$= \left. \frac{x^2}{4} \right|_0^1 = \frac{1}{4}$$

$$E[X] \Rightarrow \int \text{"}x\text{"}\, f_X(x)\, dx$$

$$= \int_x x \left[ \int_y f_{X|Y}(x|y)\, f_Y(y)\, dy \right] dx$$

$$\int_y f(x,y)\, dy = f_X(x)$$

$$E_Y\left[E[X|Y]\right] = E[X]$$

$$= \int_y \boxed{\int_x x\, f_{X|Y}(x|y)\, dx}\, f_Y(y)\, dy$$

$$\int_y \text{"}E[X|Y]\text{"}\, f_Y(y)\, dy = E_Y\left[E[X|Y]\right]$$

# Example

A stick of length one is broken at a random point, uniformly distributed over the stick. The remaining piece is broken once more.

- Find the expected length of the piece that now remains given the length of first piece
- Find the expected length of the piece

# Law of Total Expectation

$$\mathbb{E}[X] = ?$$

# Example

Consider two random variables $X$ and $Y$

- $X \sim \mathcal{N}(\mu, \sigma^2)$

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Find $\mathbb{E}[Y]$