Expectation, Higher Moments

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Q) Would you pay me Rs. 4 to pay a game where I pay you the number you get on roll of dice?

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Q) Would you pay me Rs. 4 to pay a game where I pay you the number you get on roll of dice?

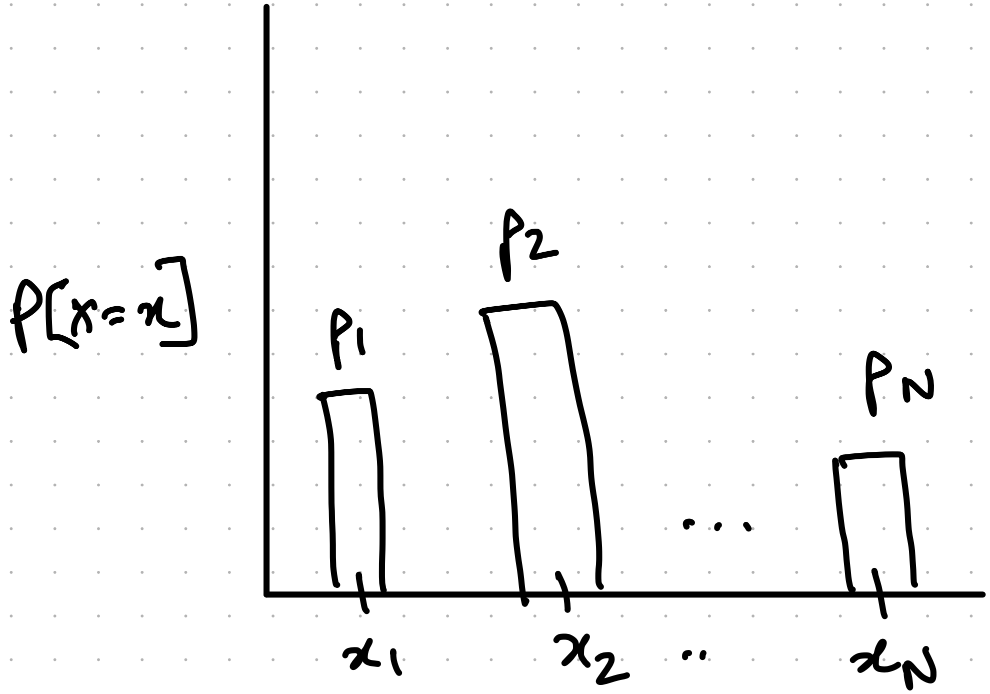
NO!

In expetation, you will lose 50 paise

$$E[X] = \sum_{x \in X(\Omega)} x \, p_X(x)$$

where

$$p_X(x) = P[X = x]$$



$$E[X] = p_1 x_1 + p_2 x_2 + \cdots p_N x_N$$

Q) COIN TOSS

- 2 coin toss
- X is. r.v. # H.

η	$X(\eta)$
TT	0
TH	1
HT	1
HH	2

$E[X] = ?$

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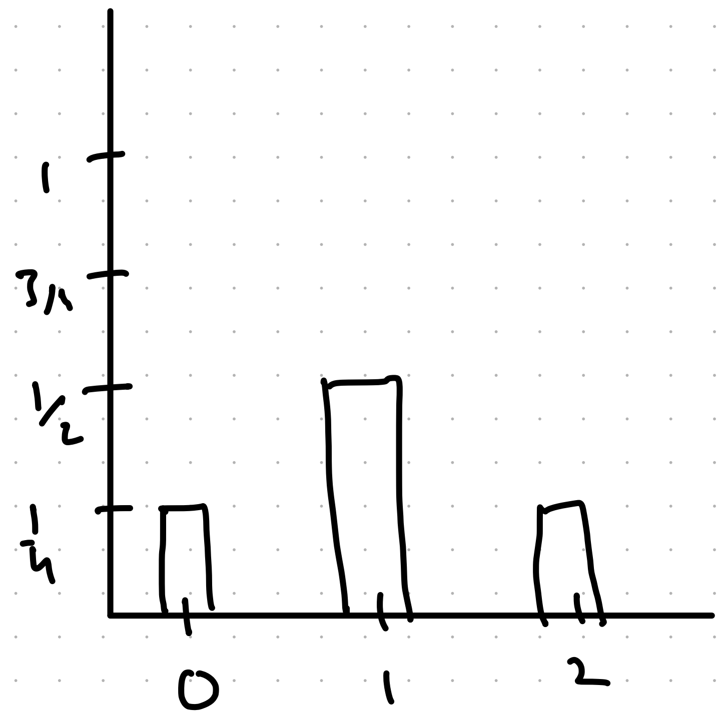

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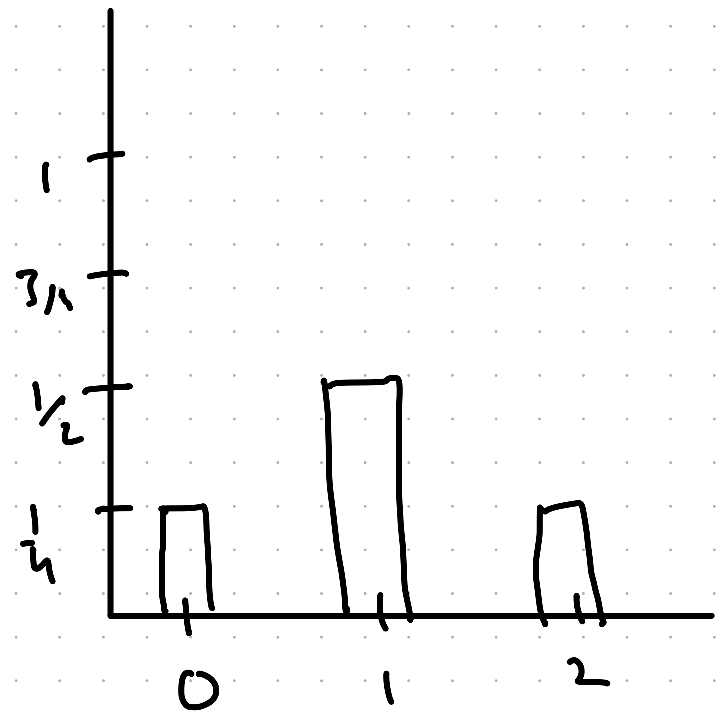

$$E[X] = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4}$$

$$= 1$$

Q) $X \sim \text{Bernoulli}(p)$

$E[X] = ?$

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$$p_X(x) = \begin{cases} p & ; x = 1 \\ 1-p & ; x = 0 \end{cases}$$

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$$E[X] = \sum_{x \in \{0,1\}} x \, p_X(x) = 0 * (1-p) + 1 * p$$

$$= p$$

Q) $X \sim \text{Binomial}(n, p)$

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$$E[X] = \sum_{k=0}^{n} k * \binom{n}{k} p^k (1-p)^{n-k}$$

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HINT: What is $\sum_{k=0}^{n} p_X(k)$?

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HINT: What is $\sum_{k=0}^{n} P_X(k)$? 1 (as PMF)

Q) $X \sim \text{Binomial}(n, p)$

$E[X] = ?$

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$E[X] = \sum_{k=0}^{n} k * \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k * \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$ [As $0 \times \ldots = 0$ so start from $k=1$]

$$= n \sum_{k=1} \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

Q) $X \sim \text{Binomial}(n, p)$

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$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$E[X] = n \sum_{k=1} \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{m=0}^{n-1} \binom{n-1}{m} p^m (1-p)^{n-1-m}$$

Q) $X \sim \text{Binomial}(n, p)$

$E[X] = ?$

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$E[X] = np \sum_{m=0}^{n-1} \binom{n-1}{m} p^m (1-p)^{n-1-m}$$

$$= np \sum p_Y(m) \qquad \text{where } Y \sim \text{Binomial}(n-1, m)$$

$$= np * 1$$

$$= np$$

Q) $X \sim \text{Poisson}(\lambda)$

$E[x] = ?$

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$$P_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

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$$= \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k e^{-\lambda}}{k!}$$ (as kn.. $= 0$)

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$$= \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k e^{-\lambda}}{k!} \quad (\text{as } k\ldots = 0)$$

$$= \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!} = \lambda \sum_{m=0}^{\infty} \frac{\lambda^m \cdot e^{-\lambda}}{m!}$$

$$= \lambda \cdot 1 = \lambda$$

Properties of Expectation

1) Function of X

For any function g;

$$E[g(X)] = \sum_x g(x) p_X(x)$$

Properties of Expectation

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For any function g;

$$E[g(X)] = \sum_x g(x)\, p_X(x)$$

Q). You toss two coins with $p = .5$

0 heads = 0 Rupees

1 heads = 10 Rupees

2 heads = 25 Rupees

game playing = 20 Rupees

On average what is your return?

Properties of Expectation

1) Function of X

For any function g;

$$E[g(X)] = \sum_x g(x)\, p_X(x)$$

Q). You toss two coins:

0 heads = 0 Rupees

1 heads = 10 Rupees

2 heads = 25 Rupees

game playing = 20 Rupees

On average what is your return?

$X \sim \text{Binomial}(2, 0.5)$

$P[X=0] = p_X(0) = \binom{2}{0}(.5)^0(.5)^{2-0}$

$= 1/4$

$p_X(1) = 1/2$

$p_X(2) = \frac{1}{4}$

Properties of Expectation

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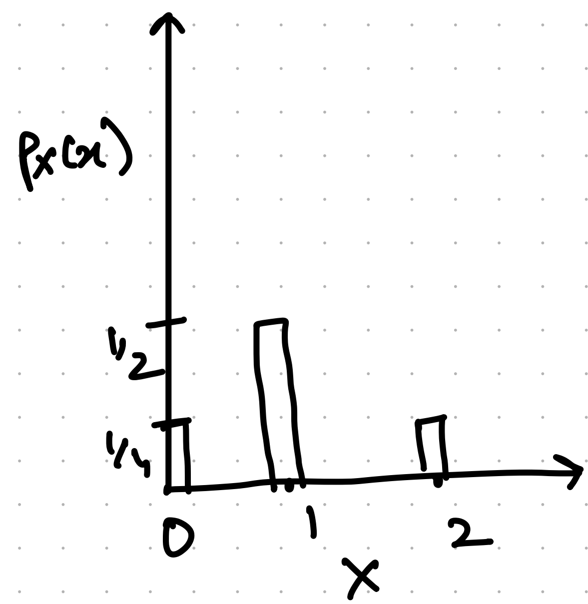

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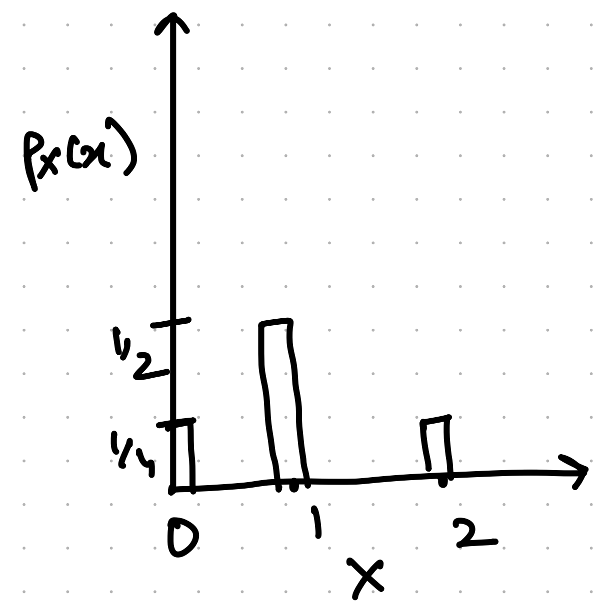

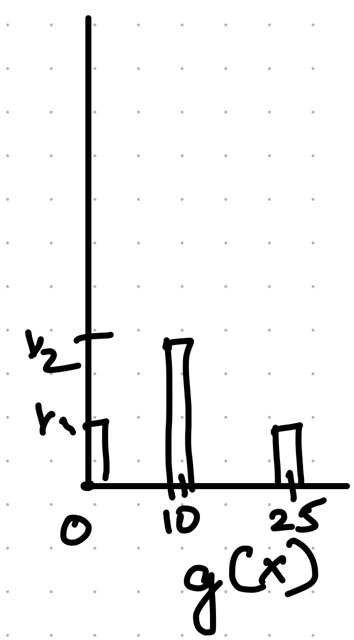

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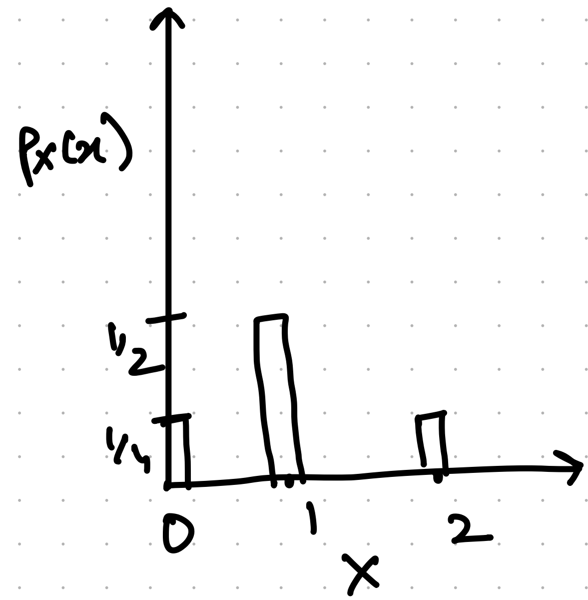

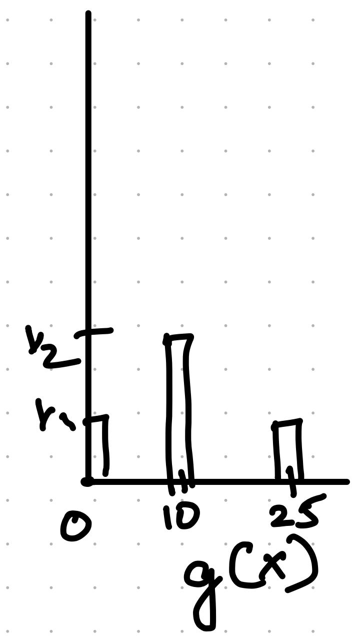

$$E[g(X)] = 0 \times \frac{1}{4} + 10 \times \frac{1}{2} + 25 \times \frac{1}{4}$$

$$= 5 + 6.25 = 11.25$$

Properties of Expectation

1) Function of X

For any function g;

$$E[g(X)] = \sum_x g(x)\, p_X(x)$$

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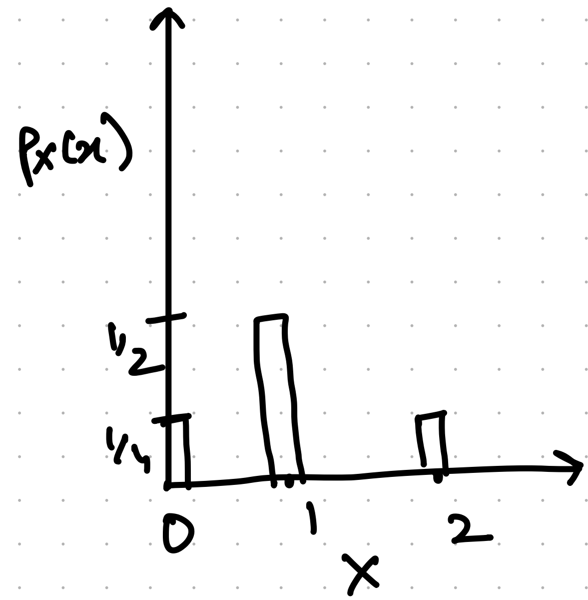

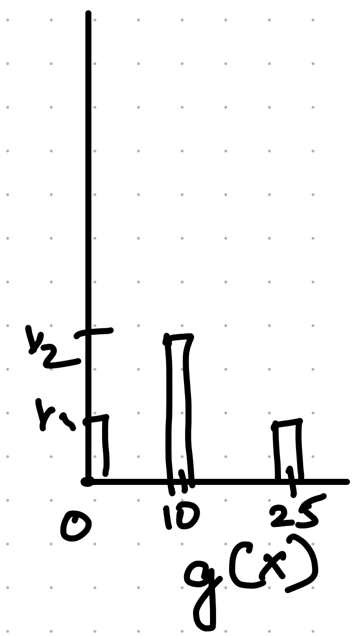

$E[g(X)] = 11.25$

Expected gain = $E[g(X)] - 20 = -8.75$

Properties of Expectation

1) Scale $E[cX] = c\,E[X]$

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$$E[cX] = \sum_x c x \, p_X(x)$$

$$= c \sum_x x \, p_X(x)$$

$$= c\,E[X]$$

Properties of Expectation

1) Scale $E[cX] = c\,E[X]$

Q) Assume $X \sim \text{POISSON}(\lambda = 5)$ = # orders you sell per day

For each sale you get 1000 INR:

what is Expected money earned daily

Properties of Expectation

ii) Scale $E[cX] = c\,E[X]$

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For each sale you get 1000 INR:

what is Expected money earned daily

Exp. Money $= E[1000X] = 1000\,E[X] = 1000 \times 5 = 5000$ INR

Properties of Expectation

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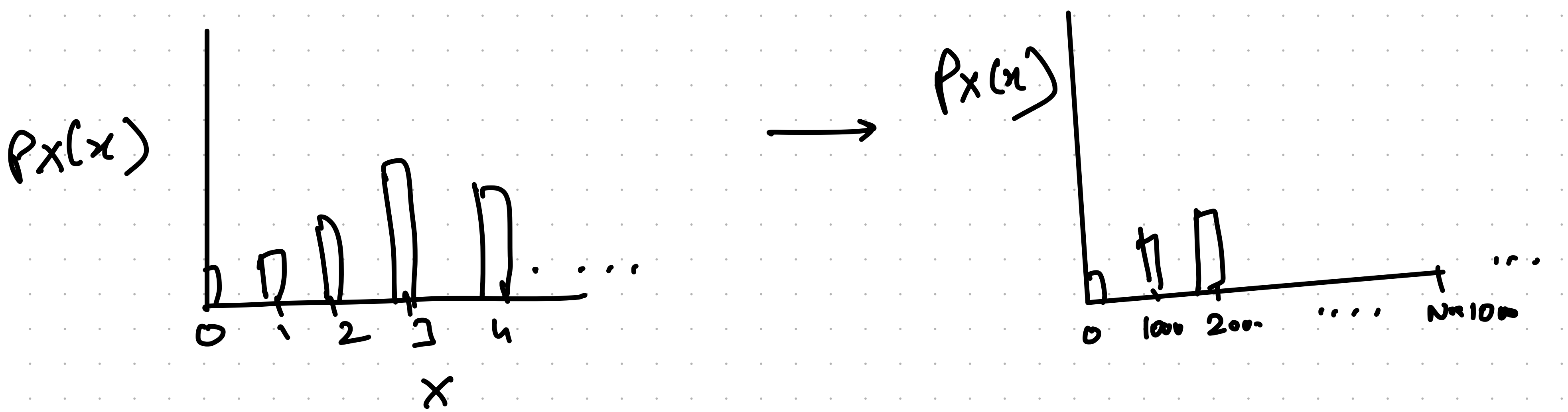

Properties of Expectation

IV) Linearity of Expectation

$$E[g(X)+h(X)] = E[g(X)] + E[h(X)]$$

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$$E[g(X)+h(X)] = E[g(X)] + E[h(X)]$$

$$= \sum_x \big(g(x) + h(x)\big)\, p_X(x)$$

$$= \sum_x g(x)\, p_X(x) + \sum_x h(x)\, p_X(x)$$

$$= E[g(X)] + E[h(X)]$$

Properties of Expectation

IV) Linearity of Expectation

$$E[g(X)+h(X)] = E[g(X)] + E[h(X)]$$

Q). $X \sim \text{POISSON}(3)$ denote r.v. for # daily sales.

$g(X) = 1000X$ (Commission per sale)

$$h(X) = \left.\begin{cases} -2000 & ; 0 \text{ sale} \\ -500 & ; 1 \text{ sale} \end{cases}\right\} \text{[Penalty]}$$

$\quad\quad\quad 0 \; ; \geq 2$ sales

what is expected amt. you take home

Properties of Expectation

IV) Linearity of Expectation

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$$\quad\quad 0 \;;\; \geq 2 \text{ sales}$$

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$$0 \quad ; \geq 2 \text{ sales}$$

$$E[g(x)+h(x)] = E[g(x)] + E[h(x)]$$

$$= 1000 * 3 + (-2000) * \frac{e^{-3} 3^{0}}{0!} + (-500) * \frac{e^{-3} 3^{1}}{1!}$$

Properties of Expectation

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$$\qquad\qquad 0 \; ; \geq 2 \text{ sales}$$

$$E[g(X)+h(X)] = E[g(X)] + E[h(X)]$$

$$= 1000 * 3 + (-2000) * \frac{e^{-3} 3^0}{0!} + (-500) * \frac{e^{-3} 3^1}{1!}$$

$$= 3000 + (-99) + (-75) \simeq 3000$$

Properties of Expectation

IV) DC shift

$$E[X+c] = E[X] + c$$

-

Properties of Expectation

IV) DC shift

$$E[X+c] = E[X] + c$$

$$= \sum_{x} (x+c)\, p_X(x)$$

$$= \sum_{x} x\, p_X(x) + c \sum_{x} p_X(x)$$

$$= E[X] + c \cdot 1$$

$$= E[X] + c$$

Q) You have option to play 2 games

G_1 : Heads +2000Rs

Tails −2000Rs

G_2 : Heads +200Rs

Tails −200Rs

what is expected return for both games?

Q) You have option to play 2 games

G_1 : Heads +2000 Rs
Tails −2000 Rs

G_2 : Heads +200 Rs
Tails −200 Rs

What is expected return for both games?

0

<u>BUT</u> G_1 is riskier : Larger Variance!

Moments

k^{th} moment of r.v. X is

$$E[X^k] = \sum_x x^k p_X(x)$$

Variance

variance of r.v. showing spread is

$$VAR[X] = E[(X - \mu_X)^2]$$

Moments

k^{th} moment of r.v. X is

$$E[X^k] = \sum_x x^k p_X(x)$$

Variance

variance of r.v. showing spread is

$$VAR[X] = E[(X - E[X])^2]$$

$$= E[X^2 - 2X E[X] + E[X]^2]$$

$$= E[X^2] - 2E[X].E[X] + E[X]^2$$

$$= E[X^2] - E[X]^2$$

Moments

k^{th} moment of r.v. X is

$$E[X^k] = \sum_x x^k \, p_X(x)$$

Variance

variance of r.v. showing spread is

$$VAR[X] = E[(X - E[X])^2]$$

$$= E[X^2 - 2X\,E[X] + E[X]^2]$$

$$= E[X^2] - 2\,E[X].E[X] + E[X]^2$$

$$= E[X^2] - E[X]^2$$

X is r.v.
$E[X]$ is a constant once computed.

$\therefore E[E[X]]$
$= E[X]$

$(\because E[c] = c)$

Q). $X \sim$ Bernoulli (p)

Find 2^{nd}, 3^{rd} moment and variance.

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Find 2^{nd}, 3^{rd} moment and variance.

$$E[X^2] = \Sigma x^2 p_X(x)$$

$$= 0^2 * (1-p) + 1^2 * (p)$$

$$= p$$

Q). $X \sim$ Bernoulli (p)

Find 2^{nd}, 3^{rd} moment and variance.

$$E[X^2] = \Sigma x^2 p_X(x)$$

$$= 0^2 * (1-p) + 1^2 * (p)$$

$$= p$$

$$VAR[X] = E[X^2] - E[X]^2$$

$$= p - p^2$$

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Find 2^{nd}, 3^{rd} moment and variance.

$$E[X^2] = \sum x^2 p_X(x)$$

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Q) For what p is VAR maximized.

$$\frac{\partial}{\partial p}(p - p^2) = 0 \Rightarrow 1 - 2p = 0$$

$$\Rightarrow p = 0.5$$

Q). $X \sim \text{Bernoulli}(p)$

$$\text{VAR}[X] = p - p^2$$

Q) For what p is VAR maximized.

$$\frac{\partial}{\partial p}(p - p^2) = 0 \Rightarrow 1 - 2p = 0$$

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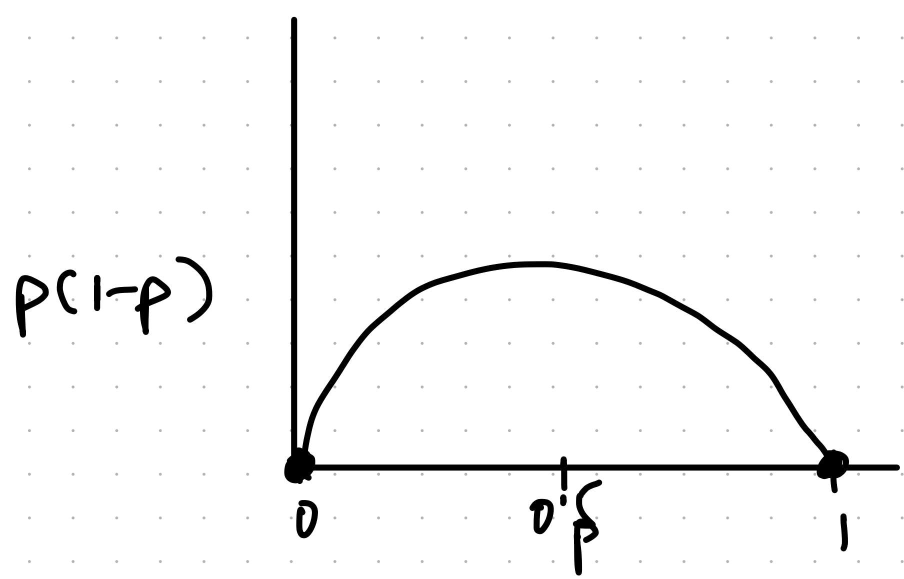

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$$E[X] = np$$

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$$= \sum_{k=1}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k^2 \cdot \frac{n!}{(n-k)!\, k.(k-1)!} \cdot p^k (1-p)^{n-k}$$

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$$= \sum_{k=1}^{n} (k^2 + k - k) \frac{n!}{(n-k)!\, k(k-1)!} \cdot p^k (1-p)^{n-k}$$

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$$= \sum_{k=1}^{n} \frac{k(k-1)\, n!\, p^k (1-p)^{n-k}}{(n-k)!.\, k(k-1)!} + \sum_{k=1}^{n} k \frac{n!}{(n-k)!\, k!} p^k (1-p)^{n-k}$$

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$$= \sum_{k=2}^{n} \frac{n!\, p^k (1-p)^{n-k}}{(n-k)!\, (k-2)!} + E[X]$$

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$$= \sum_{k=2}^{n} \frac{n!\, p^k (1-p)^{n-k}}{(n-k)!\,(k-2)!} + E[X]$$

$$= \sum_{k=2}^{n} n(n-1) \frac{(n-2)!\, p^k (1-p)^{n-k}}{(n-k)!\,(k-2)!} + E[X]$$

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Now:, $$E[X^2] = \sum_{k=2}^{n} n(n-1) \frac{(n-2)!\, p^k (1-p)^{n-k}}{(n-k)!\,(k-2)!} + E[X]$$

$$= \sum_{k=2}^{n} n(n-1) \frac{(n-2)!\, p^{k-2} \cdot (1-p)^{n-k}}{(n-k)!\,(k-2)!} \cdot p^2 + E[X]$$

$$= p^2 n(n-1) \sum_{k=2}^{n} \frac{(n-2)!\, p^{k-2} (1-p)^{n-k}}{(n-k)!\,(k-2)!} + E[X]$$

let $l = k-2$

$k_{max} = n$

$k_{min} = 2$

$l_{min} = 0$

$l_{max} = n-2 = m$

Q) $X \sim \text{Binomial}(n, p)$

Now:, $$E[X^2] = p^2 n(n-1) \sum_{l=0}^{m} \frac{m!\, p^l (1-p)^{m-l}}{(m-l)!\, l!} + E[X]$$

$$= p^2 n(n-1) \sum_{l=0}^{m} \text{Binomial}(m, l) + E[X]$$

$$= p^2 n(n-1) * 1 + np$$

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$$= p^2 n(n-1) * 1 + np$$

$$\text{VAR}(X) = E[X^2] - E[X]^2$$

$$= p^2 n(n-1) + np - (np)^2$$

$$= p^2 n^2 - p^2 n + pn - n^2 p^2$$

$$\text{VAR}(X) = np(1-p)$$

Properties of Variance

1) Scale

$$VAR[cX] = c^2 \, VAR[X]$$

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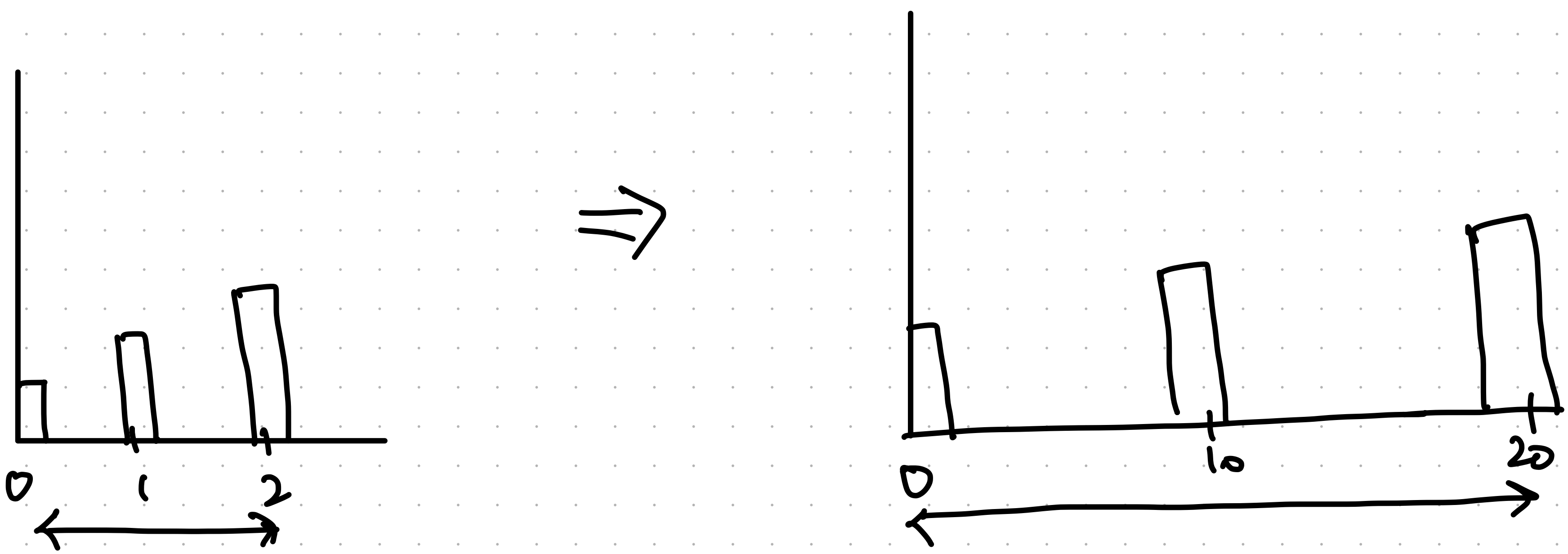

Properties of Variance

II) DC shift

$$VAR[X+c] = VAR[X]$$

Properties of Variance

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$$VAR[X+c] = VAR[X]$$

$$= E[(X+c)^2] - (E[X+c])^2$$

$$= E[X^2+c^2+2cX] - (E[X]+c)^2$$

$$= E[X^2] + c^2 + 2cE[X] - E[X]^2 - c^2 - 2cE[X]$$

$$= E[X^2] - E[X]^2$$

$$= VAR[X]$$

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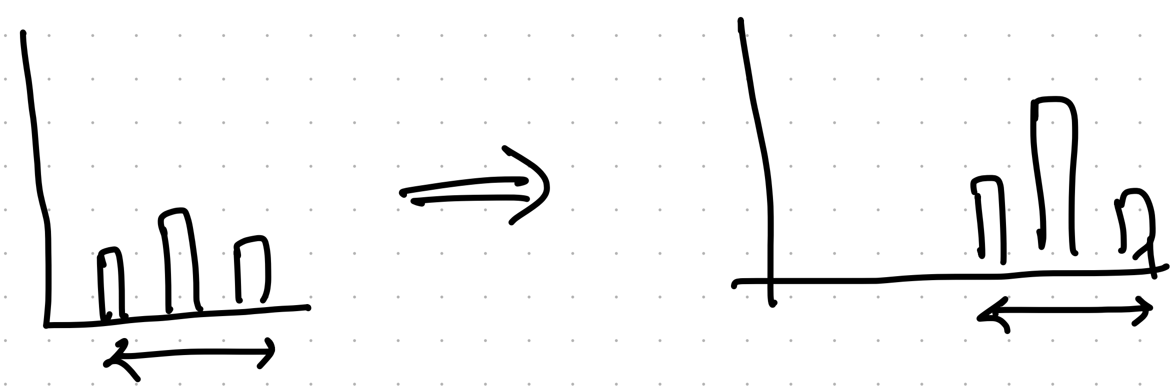

Revisiting +2000, -2000 game

Let us say win amt = q ; Loss amt = $-q$

Revisiting +2000, -2000 game

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ASIDE

If some r.v. $X \sim$ Binomial (N, p)

$$E[X] = Np$$

$$VAR[X] = Np(1-p)$$

$$\propto N$$

If some r.v. $X \sim$ Binomial (N, p)

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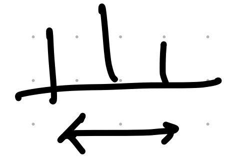
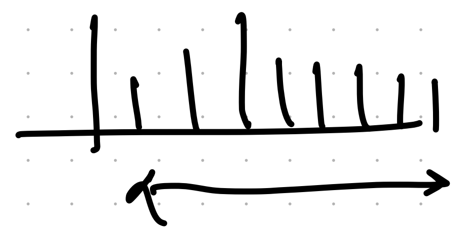
$N = 1$; support $\{0, 1\}$

$N = 2$; support $\{0, 1, 2\}$



$\vdots$

$N = N$; support $\{0, 1, \ldots, N\}$



Revisiting +2000, -2000 game

Let us say win amt = q Loss amt = $-q$

let us use a r.v W denoting # wins in N rounds.

$$W \sim \text{Binomial}(N, p)$$

Revisiting +2000, -2000 game

Let us say win amt = q Loss amt = $-q$

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(heads)

$W \sim \text{Binomial}(N, p)$

r.v. L denoting # Losses (tails)

$L = N - W$

Revisiting +2000, -2000 game

Let us say win amt = q Loss amt = $-q$

let us use a r.v W denoting # wins (heads) in N rounds.

$$W \sim \text{Binomial}(N, p)$$

r.v. L denoting # Losses (tails)

$$L = N - W$$

Total reward after N rounds

$$S_N = qW - qL = qW - q(N-W)$$

$$= 2qW - qN$$

Revisiting +2000, -2000 game

Total reward after N rounds

$$S_N = 2qW - qN$$

Now, $E[S_N] = E[2qW] - E[qN]$

Revisiting +2000, -2000 game

Total reward after N rounds

$$S_N = 2qW - qN$$

Now, $E[S_N] = E[2qW] - E[qN]$

$$= 2q\,E[W] - qN$$

$$= 2q * N * p - qN$$

$$= qN(2p - 1)$$

Revisiting +2000, -2000 game

Total reward after N rounds

$$S_N = 2qW - qN$$

Now, $E[S_N] = E[2qW] - E[qN]$

$$= 2q\,E[W] - qN$$

$$= 2q * N * p - qN$$

$$= qN(2p - 1)$$

For $p = 0.5$

$$E[S_N] = qN(2 * .5 - 1) = 0$$

Revisiting +2000, -2000 game

Total reward after N rounds

$$S_N = 2qW - qN$$

$$VAR[S_N] = VAR[2qW - qN]$$

Revisiting +2000, -2000 game

Total reward after N rounds

$$S_N = 2qW - qN$$

$$VAR[S_N] = VAR[2qW - \underbrace{qN}_{\text{constant}}]$$

$$= VAR[2qW]$$

Revisiting +2000, -2000 game

Total reward after N rounds

$$S_N = 2qW - qN$$

$$\mathrm{VAR}[S_N] = \mathrm{VAR}[2qW - qN]$$

constant

$$= \mathrm{VAR}[2qW]$$

$$= 4q^2 \, \mathrm{VAR}[W]$$

$$= 4q^2 * N * p * (1-p)$$

$$\mathrm{VAR}[S_N] = 4q^2 N * \frac{1}{2} * \frac{1}{2} = q^2 N$$

Revisiting +2000, -2000 game

Total reward after N rounds

$$S_N = 2qW - qN$$

$$E[S_N] = 0$$

$$VAR[S_N] = 4q^2 N * \frac{1}{2} * \frac{1}{2} = q^2 N$$

Revisiting +2000, -2000 game

Total reward after N rounds

$$S_N = 2qW - qN$$

$$E[S_N] = 0$$

$$VAR[S_N] = 4q^2 N * \frac{1}{2} * \frac{1}{2} = q^2 N$$

$$VAR[S_N] \propto N$$

As $N \uparrow$; $VAR[S_N] \uparrow$