Law of Large Numbers

and

Central Limit Theorem

(Intro)

and why we need to study

1. Markov's Inequality
2. Chebyshev's Inequality
3. Moment generating functions

Population v/s Sample

Q) what is the avg height of a person in India?

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Impossible to measure

Population mean = True average height $= \mu$

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Sample : Randomly select 500 people and measure height

Sample height = $\bar{X}_{500}$

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Population mean = True average height = μ

Sample : Randomly select 500 people and measure height

Sample height = $\bar{X}_{500}$

Q) What is relationship b/w $\bar{X}_n$ and μ?

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From a sample

Imagine flipping fair coin many times (N)

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What do you <u>expect</u> $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{N} x_i$

to converge to?

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(N)

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Sample average converges to population mean
(Law of large numbers)

Say person 1 flipped 'N' coins and got $\bar{X}_{n,1}$

" " 2 . . . . $\bar{X}_{n,2}$

$\vdots$

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Q: what is distribution of $\bar{X}_{n,i}$

$$\bar{X}_n \sim N(\dots, \dots)$$

Markov's Inequality

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Assume $E[X] = 4$

What is $P[X \geq 20]$?

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3. We know $X \geq 0$

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(slides ..)

Markov's Inequality

Say X = # days a patient stays at a hospital

Assume $E[X] = 4$

what is $P[X \geq 20]$?

$P[X \geq 20]$; $u = 20$

$$P[X \geq u] \leq \frac{E[x]}{u}$$

$$\therefore P[X \geq 20] \leq \frac{4}{20} \leq .2$$

Q) Total wealth of 10 companies is Rs 10,000

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P[wealth of a company $\geq$ 5000]

$E[X] = 1,000$

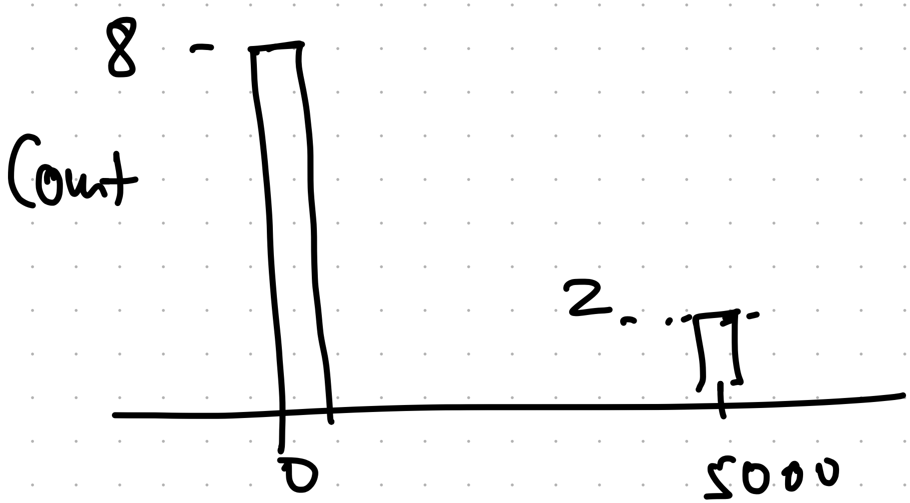
$$P[W \geq 5000] \leq \frac{1000}{5000}$$

$$\leq .2$$

Q) Total wealth of 10 companies is Rs 10,000

P[wealth of a company ≥ 5000]

Consider 8 companies wealth 0; remaining two wealth 5000



$P[w \geq 5000] = .2$

$\leq \frac{1000}{5000}$

$\leq .2$

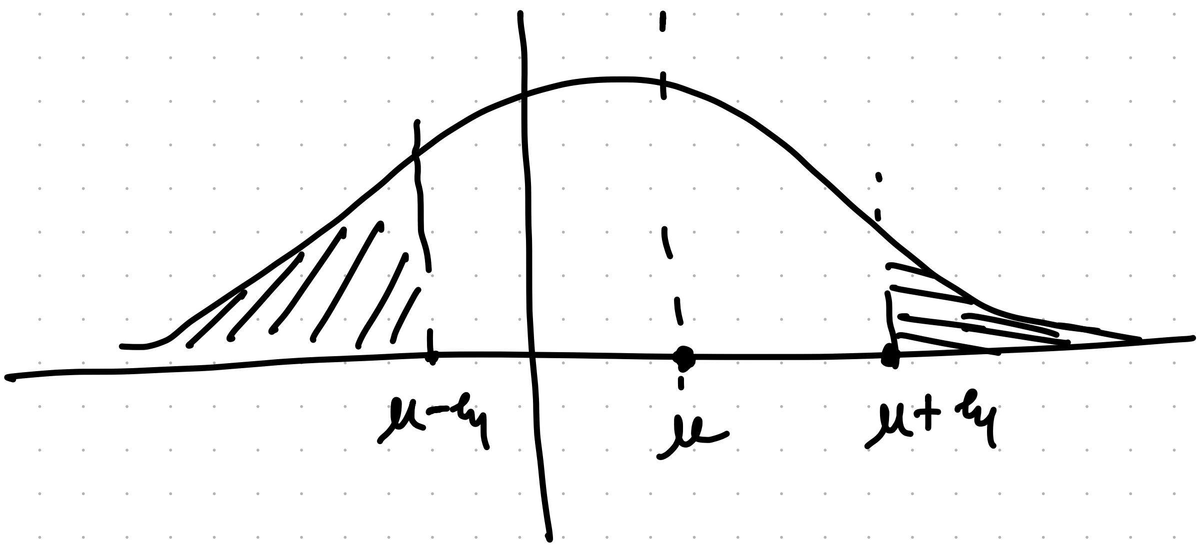
Tight bound!

Q: when does tight bound manifest?

Chebyshev's Inequality

X is r.v. with mean μ:

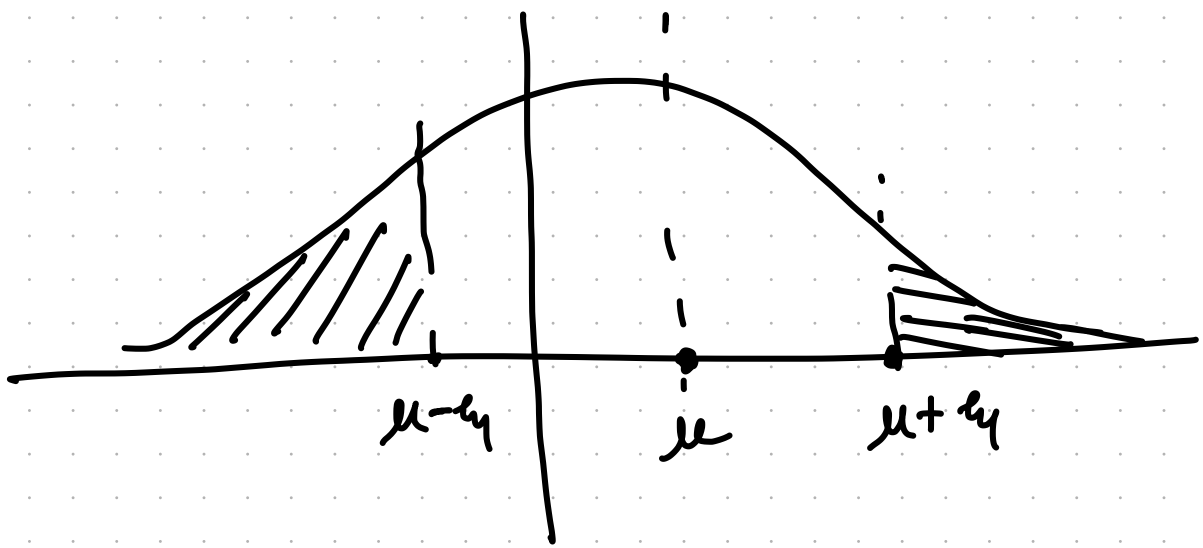
$$P\left[|X-\mu| \geq \varepsilon\right] \leq \frac{Var[X]}{\varepsilon^2}$$



Chebyshev's Inequality

X is r.v. with mean μ:

$$P[|X-\mu| \geq k]$$



$$P[|X-\mu| \geq k] = P[(X-\mu)^2 \geq k^2]$$ ($\because$ for $x \geq 0$: x^2 is monotonically increasing)

Chebyshev's Inequality

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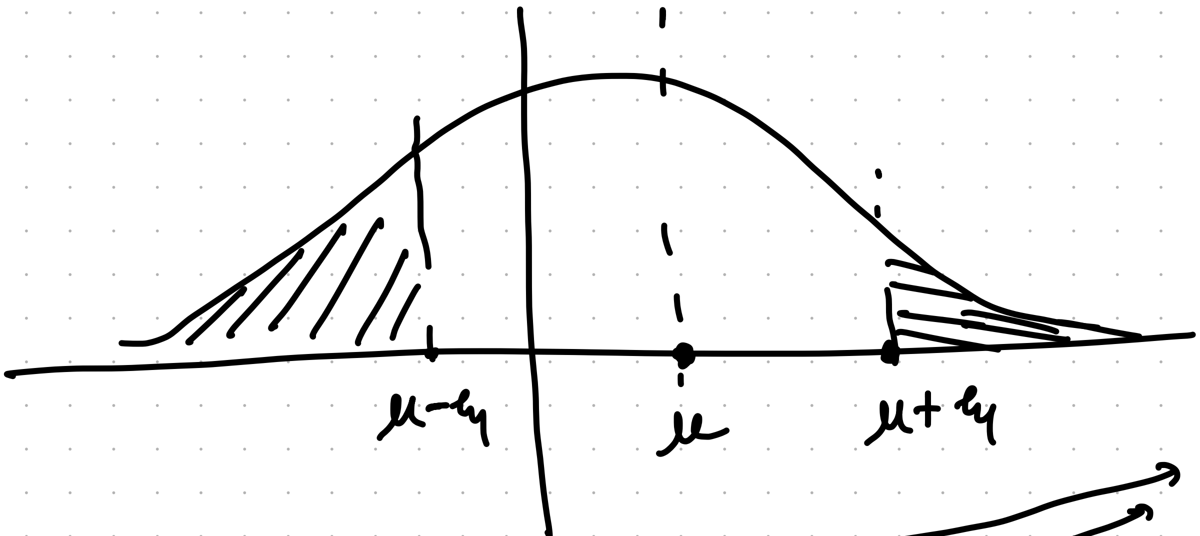

Both events are equivalent

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($\because$ for $x \geq 0$: x^2 is monotonically increasing)

Chebyshev's Inequality

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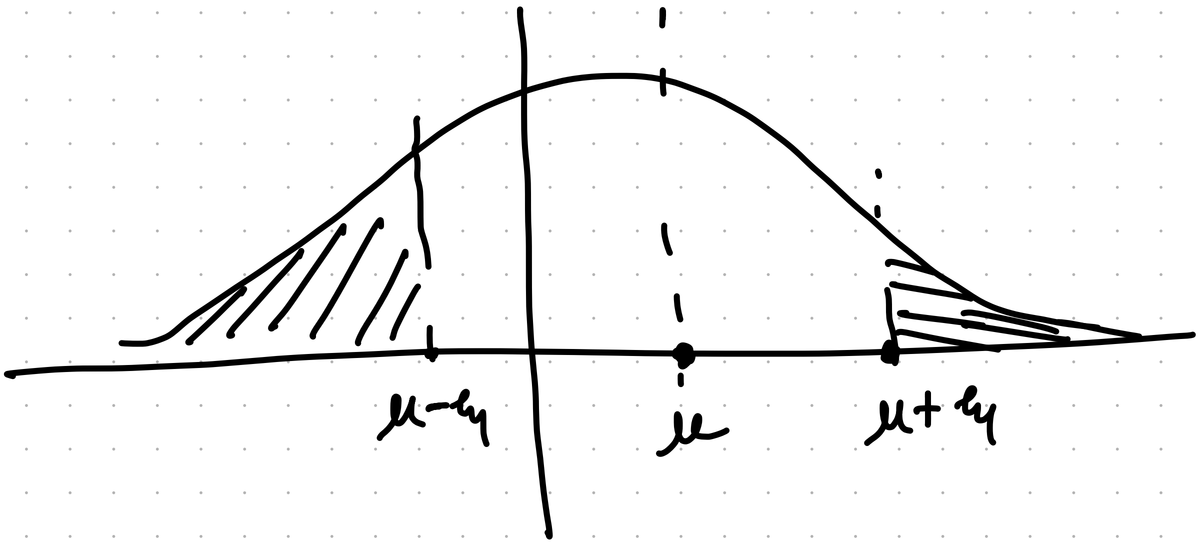

$$P[|X-\mu| \geq k] = P[(X-\mu)^2 \geq k^2]$$

Let $Y = (X-\mu)^2$; $b = k^2$

Chebyshev's Inequality

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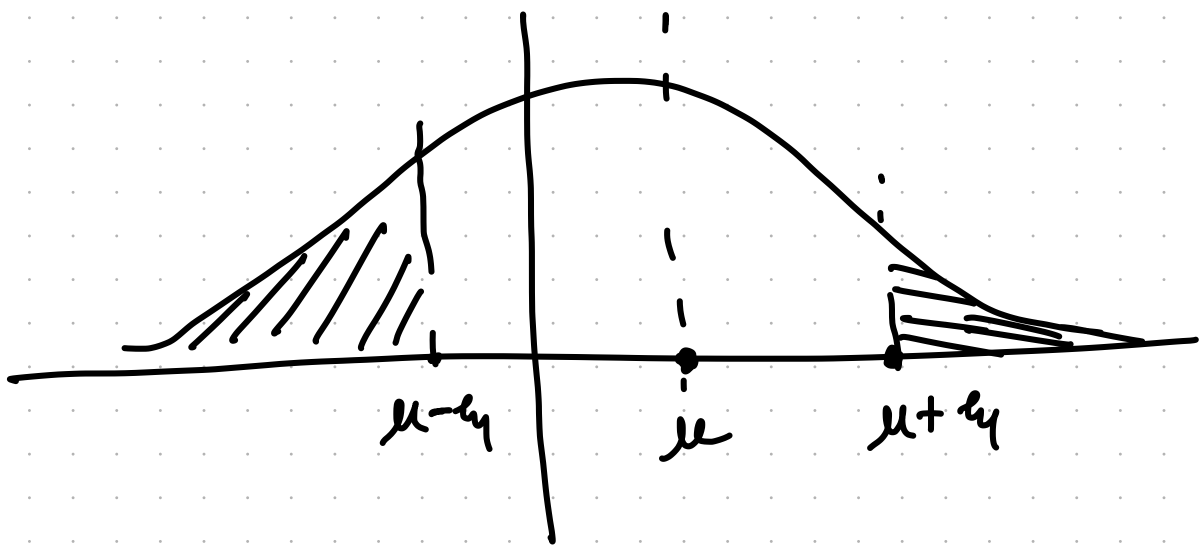


$$P[\,|X-\mu| \geq \epsilon\,] = P[(X-\mu)^2 \geq \epsilon^2]$$

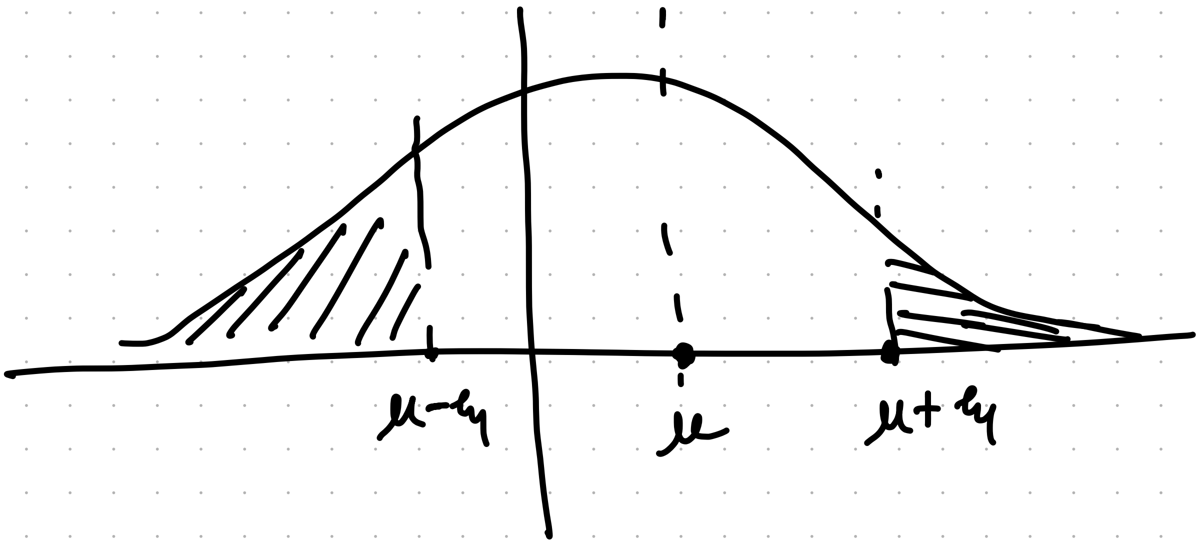
Let $Y = (X-\mu)^2$; $b = \epsilon^2$; $Y \geq 0$

$$P[Y \geq b] \leq \frac{E[Y]}{b}$$

Chebyshev's Inequality

X is r.v. with mean μ:

$$P[\,|X-\mu| \geq k\,]$$



$$P[\,|X-\mu| \geq k\,] = P[\,(X-\mu)^2 \geq k^2\,]$$

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$$E[Y] = E[(X-\mu)^2]$$
$$= VAR(X)$$

Chebyshev's Inequality

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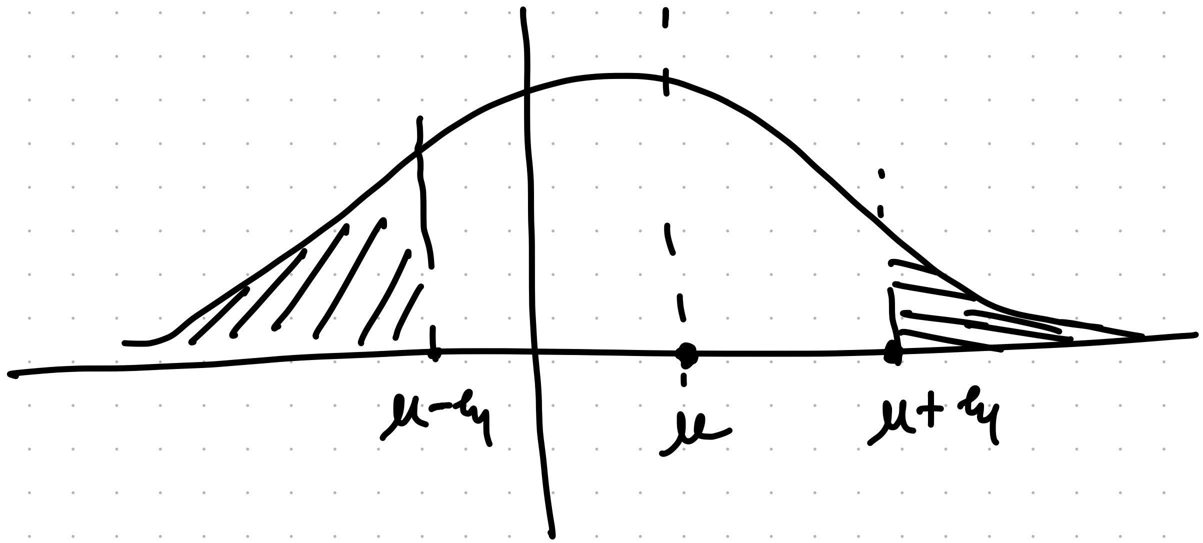
$$P[|X-\mu| \geq k]$$



$$P[|X-\mu| \geq k] = P[(X-\mu)^2 \geq k^2]$$

$$\leq \frac{VAR[X]}{k^2}$$

$$E[Y] = E[(X-\mu)^2]$$
$$= VAR[X]$$

Jensen's Inequality

(video)

$$f(E[X]) \leq E[f(X)] \quad \text{if } f \text{ convex}$$

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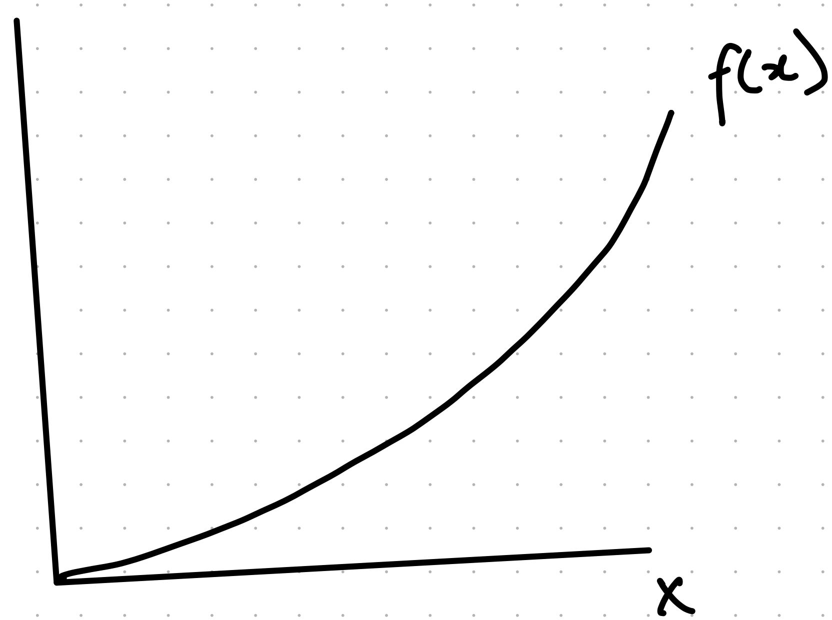
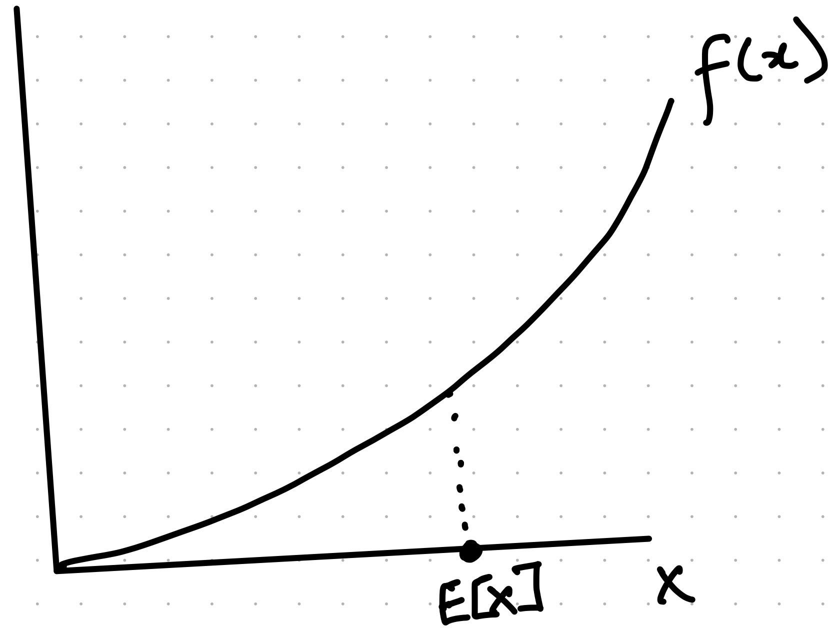


$f(E[X]) \leq E[f(X)]$ if f convex



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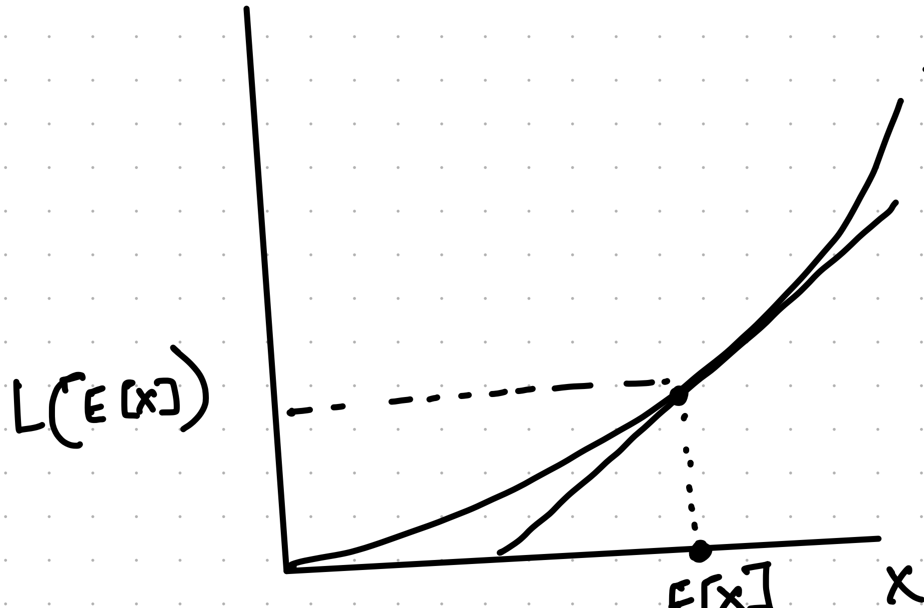

(Tangent at $\langle E[X], g(E[X]) \rangle$)

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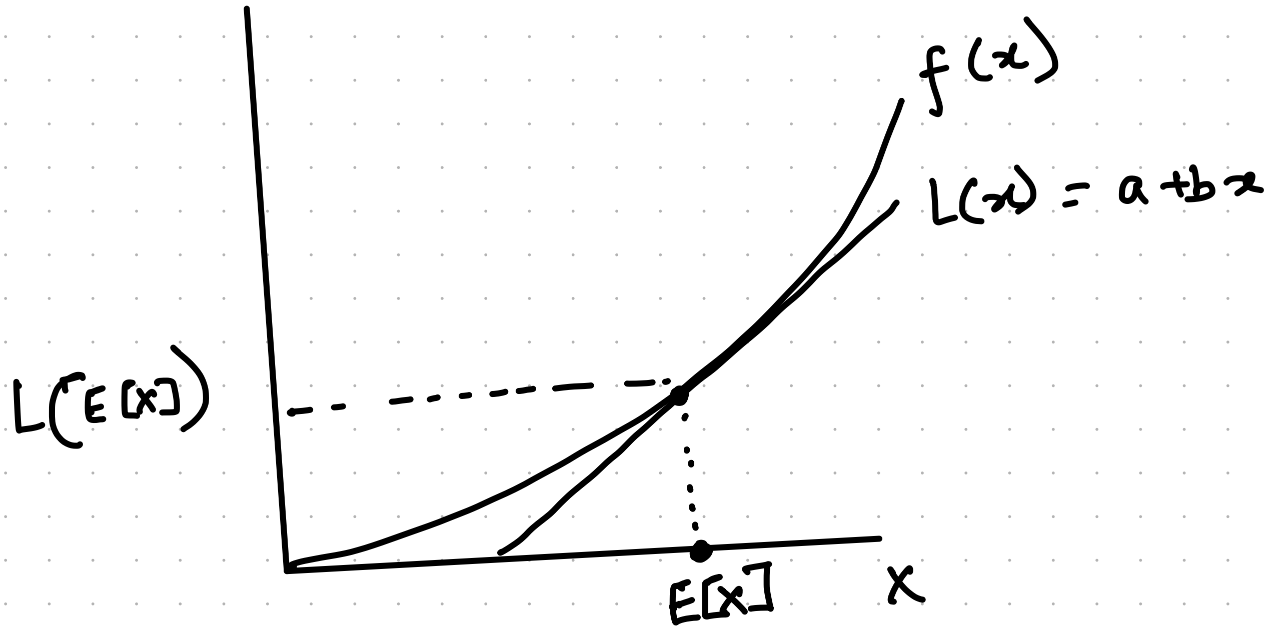

$L(E[X]) = f(E[X])$ as both intersect at $x = E[X]$

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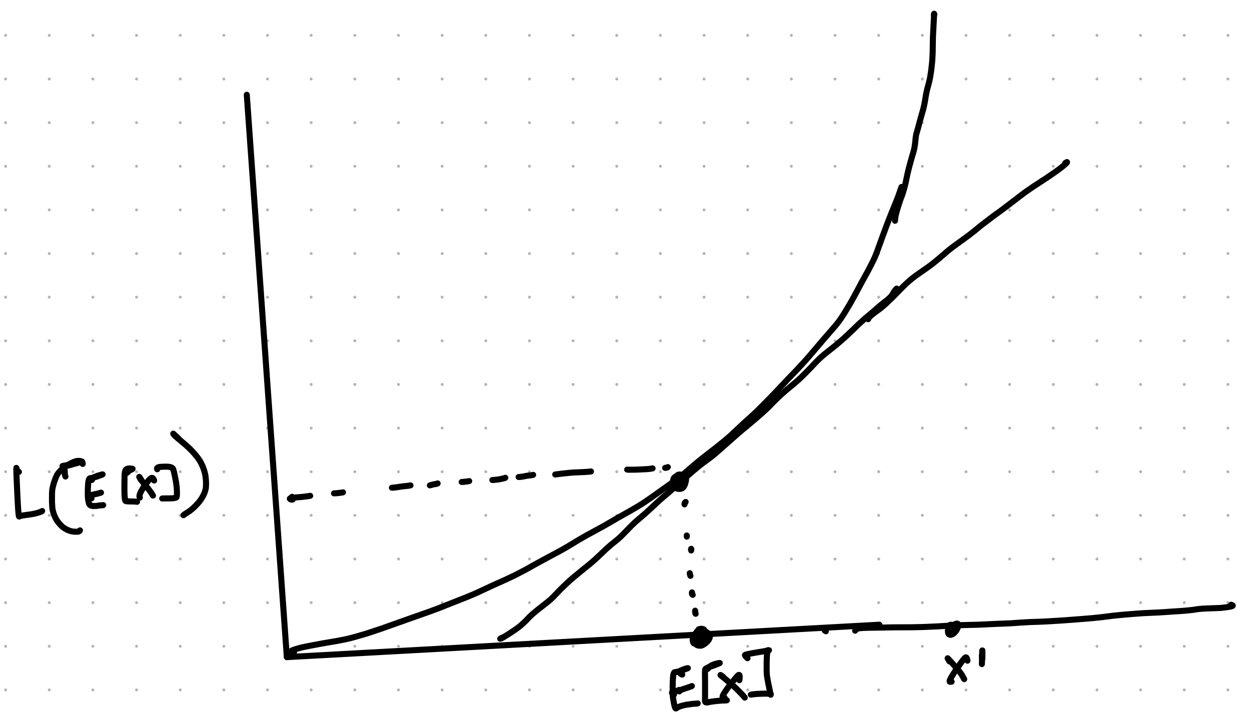

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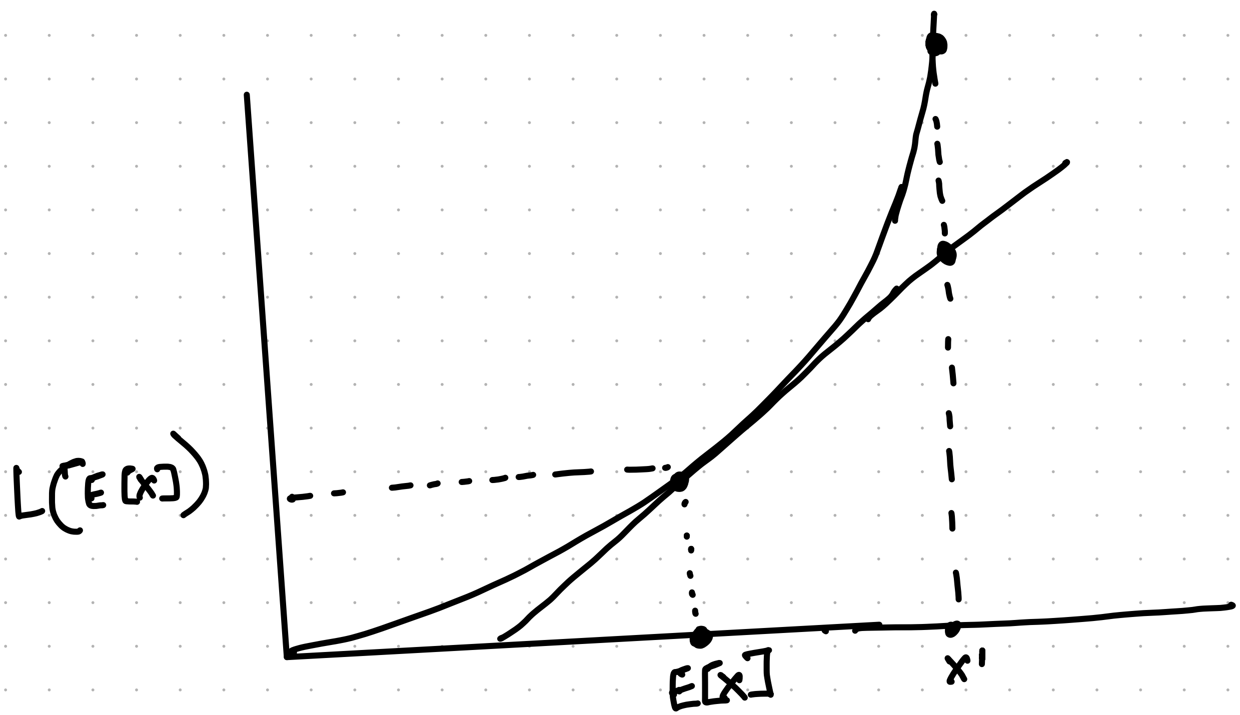

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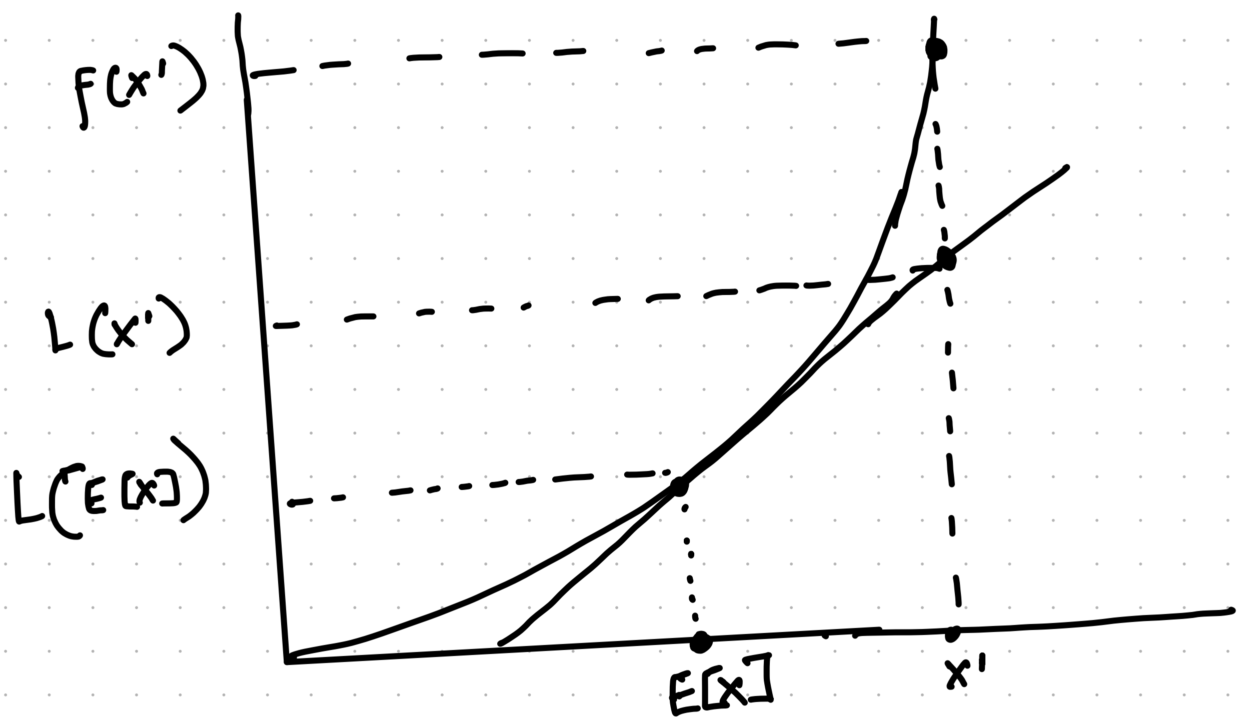
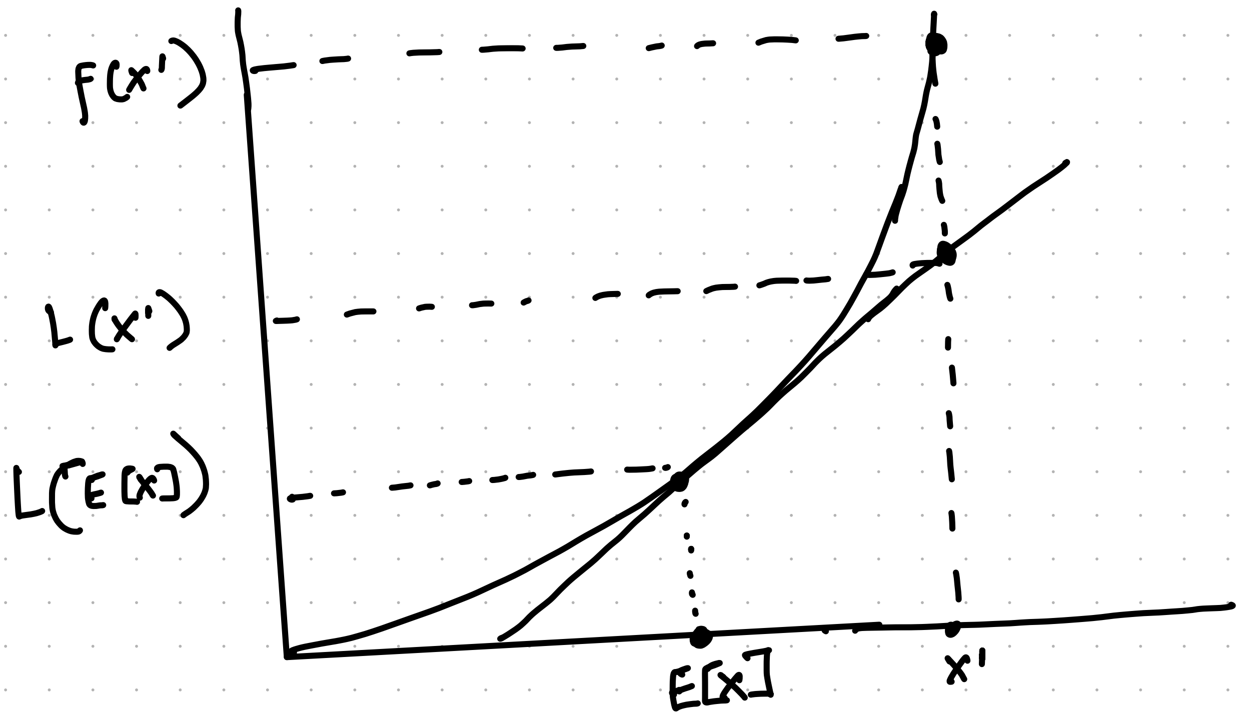


$L(E[x]) = f(E[x])$ as both intersect at $x = E[x]$

Clearly $f(x') \geq L(x')$

$$f(E[x]) \le E[f(x)] \quad \text{if } f \text{ convex}$$



$L(E[x]) = f(E[x])$ as both intersect at $x = E[x]$

$E[f(x)] \geq E[L(x)]$ as $f(x) \geq L(x) \ \forall x$

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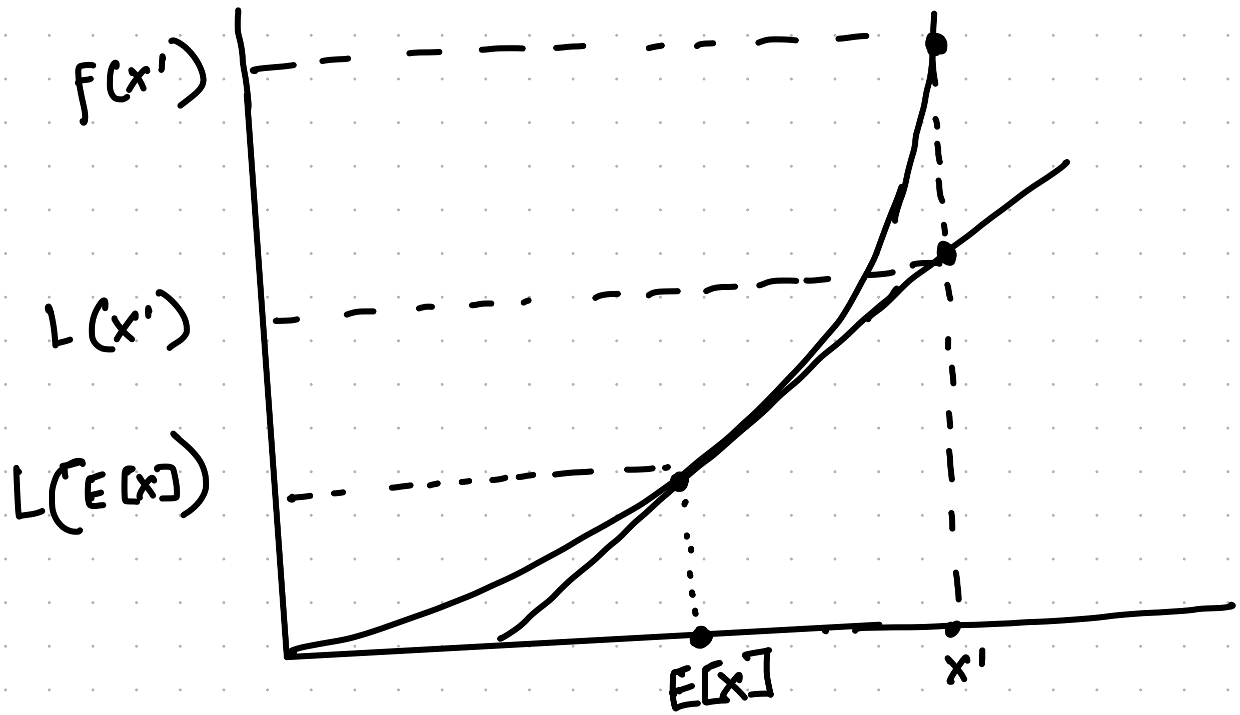

$\underline{L}(\underline{E[x]}) = \underline{f}(\underline{E[x]})$ as both intersect at $x = E[x]$

$$\begin{aligned} E[f(x)] &\geq E[L(x)] \quad \text{as } f(x) \geq L(x) \ \forall x \\ &\geq E[a + bx] \\ &\geq a + b\,E[x] \end{aligned}$$

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$$\geqslant E[a + bx]$$

$$\geqslant a + b\,E[x]$$

But $a + bE[x]$ is $L(E[x])$

$$\therefore \ E[f(x)] \geqslant L(E[x])$$

$$f(E[X]) \leq E[f(X)] \quad \text{if } f \text{ convex}$$

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But $L(E[x]) = f(E[x])$

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