Sum of Random Variables

If $X \sim N(0,1)$

$Y \sim N(0,2)$

what is distribution of $S = X + Y$

let X and Y be two <u>independent</u> rvs

with PDF $f_X(x)$ and $f_Y(y)$

let $S = X + Y$

PDF of S is:

$$f_S(s) = \underbrace{(f_X * f_Y)}_{\text{convolution}}(s) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(s-x)\, dx$$

OR FOR DISCRETE

$$P_S(s) = (P_X * P_Y)(s) = \sum_{x \in \Omega(X)} P_X(x) \cdot P_Y(s-x)$$

Proof

Starting with CDF

$$F_S(s) = P[S \leq s]$$

$$= P[X + Y \leq s]$$

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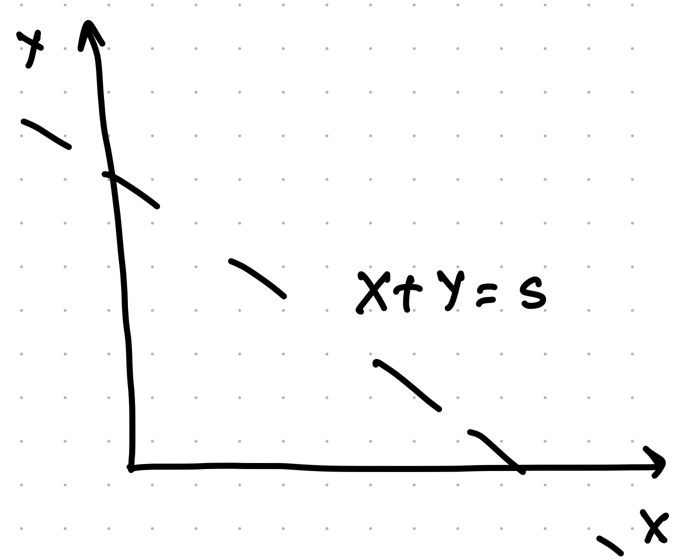

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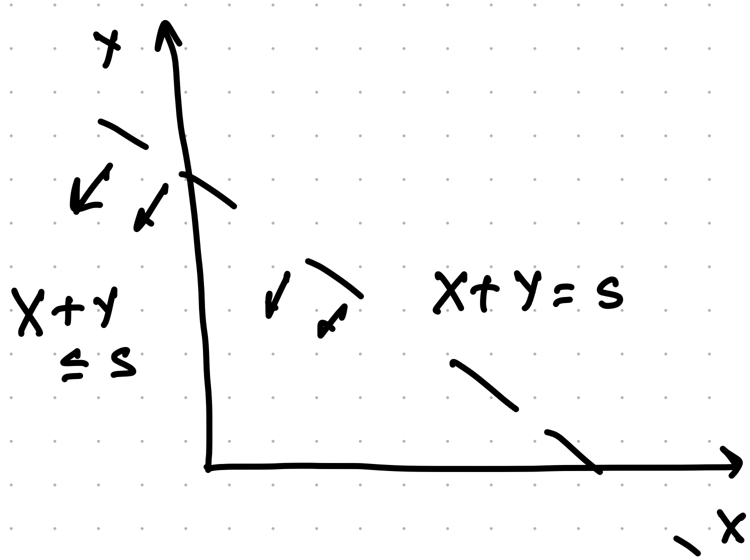
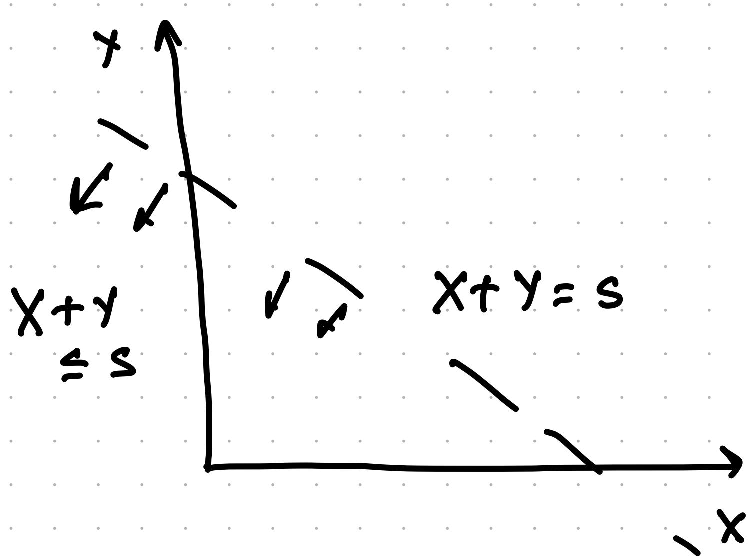


Proof

Starting with CDF

$$F_S(s) = P[S \le s]$$

$$= P[X + Y \le s] = \int_{?}^{?} \int_{?}^{?} f_{X,Y}(x,y)\,dx\,dy$$



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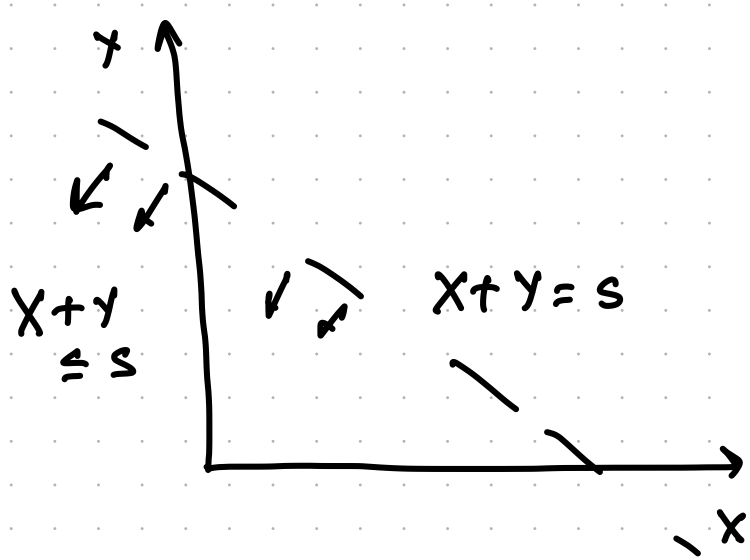

$$= \int_{?}^{?} \int_{?}^{?} f_X(x) \cdot f_Y(y)\, dx\, dy$$

(as X & Y are independent)

Proof

Starting with CDF

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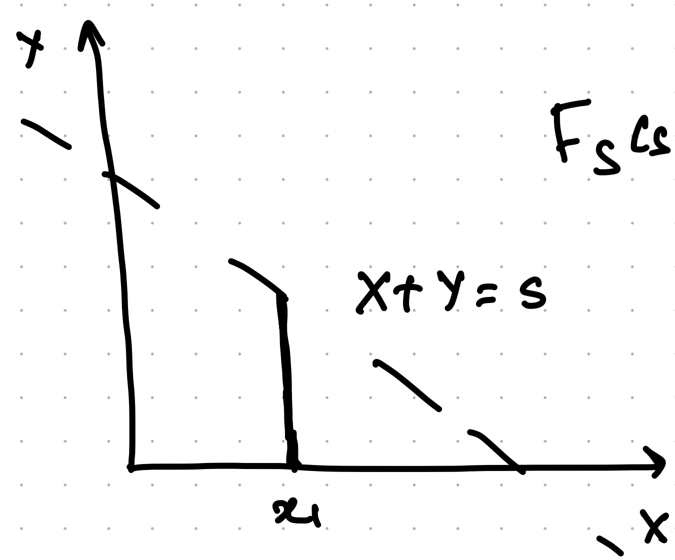

$$F_S(s) = \int_?^? \int_?^? f_X(x) \cdot f_Y(y)\, dx\, dy$$

Assume we fixed X= x1 and integrate over y

Proof

Starting with CDF

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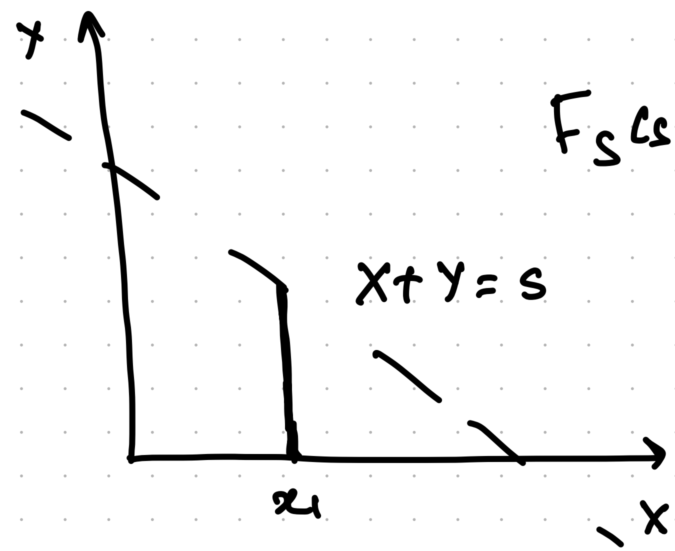

$$F_S(s) = \int_{?}^{?} \int_{?}^{?} f_X(x) \cdot f_Y(y)\, dx\, dy$$

Assume we fixed $X = x_1$ and integrate over y (first)

$$F_S(s) = \int_{-\infty}^{\infty} \int_{-\infty}^{s-x} f_X(x) \cdot f_Y(y)\, dy\, dx$$

Proof

Starting with CDF

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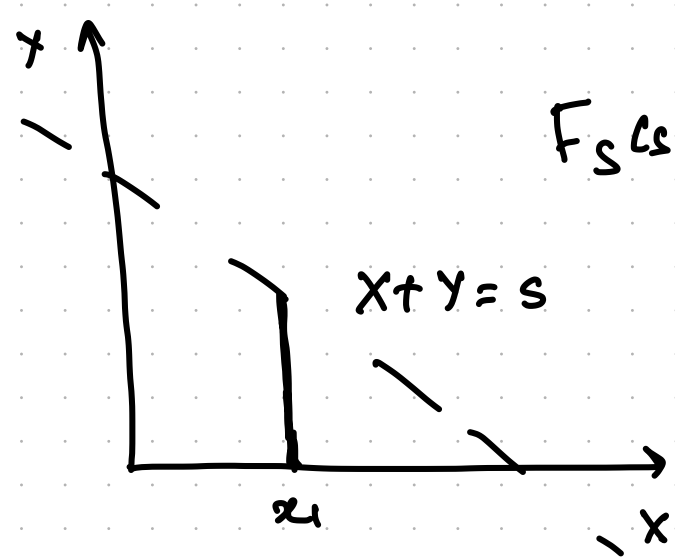

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lt for y

lt for x

Proof

Starting with CDF

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lt for x

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$$F_S(s) = P[S \leq s]$$

$$F_S(s) = \int_{-\infty}^{\infty} \int_{-\infty}^{s-x} f_X(x) \cdot f_Y(y) \, dy \, dx$$

lt. for y

lt for x

$$f_S(s) = \frac{d}{ds} F_S(s)$$

Proof

Starting with CDF

$$F_S(s) = P[S \leq s]$$

$$F_S(s) = \int_{-\infty}^{\infty} \int_{-\infty}^{s-x} f_X(x) \cdot f_Y(y) \, dy \, dx$$

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Aside : FTC

Q). let $f(s) = s^2$

$$F(t) = \int_a^t f(s)\,ds$$

Find $F'(t)$ or $\frac{d}{dt}\int_a^t f(s)\,ds$

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$$F(t) = \frac{s^3}{3}\Bigg|_a^t = \frac{t^3}{3} - \frac{a^3}{3}$$

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$$\frac{d}{dt}F(t) = F'(t) = \frac{3t^2}{3} = t^2 = f(t)$$

$$\boxed{\frac{d}{dt}\int_a^t f(s)\,ds = f(t)}$$

(N.B. Not a proof)

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$$\boxed{f_S(s) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(s-x) \, dx}$$

Q) let X and Y be independent

$$f_X(x) = \begin{cases} x e^{-x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$$

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Find PDF of $S = X + Y$

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Q) Is it possible to get $X < 0$; $Y < 0$

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Hint: Rewrite limits knowing

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$$f_S(s) = \int_{x_{MIN}}^{x_{MAX}} f_X(x) \cdot f_Y(s-x)\, dx$$

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Find PDF of $S = X + Y$

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$x_{MIN} = 0$

$x_{MAX} = S$

$(\because y \geq 0)$

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$$= \int_0^s x e^{-x} \cdot (s-x) \cdot e^{-(s-x)}\, dx \quad ; s \geq 0$$

$$= \int_0^s (xs - x^2) \cdot e^{-s}\, dx \quad ; s \geq 0$$

$$= e^{-s} \cdot \left. \frac{sx^2}{2} - \frac{x^3}{3} \right|_0^s = e^{-s} \left(\frac{3s^3 - 2s^3}{6} \right) ; s \geq 0$$

Q) let $X \sim N(\mu_1, \sigma)$; $Y \sim N(\mu_2, \sigma)$

$f_S(s) = ?$; $S = X + Y$

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$$= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{\frac{-1}{2\sigma^2}\left[(x-\mu_1)^2 + (s-x-\mu_2)^2\right]} . dx$$

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Consider $q = (x-\mu_1)^2 + (s-x-\mu_2)^2$

MAIN IDEA

Collect terms :

- quadratic in x
- linear in x
- Independent of x

Q) let $X \sim N(\mu_1, \sigma)$; $Y \sim N(\mu_2, \sigma)$

$$f_S(s) = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{\frac{-1}{2\sigma^2}\left[(x-\mu_1)^2 + (s-x-\mu_2)^2\right]} . dx$$

Consider $q = (x-\mu_1)^2 + (s-x-\mu_2)^2$

$$= x^2 + \mu_1^2 - 2x\mu_1 + s^2 + x^2 + \mu_2^2 - 2sx - 2\mu_2 s + 2x\mu_2$$

$$= 2x^2 - 2x(\mu_1 + s - \mu_2) + \mu_1^2 + \mu_2^2 + s^2 - 2\mu_2 s$$

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$$= 2x^2 - 2x(\mu_1 + s - \mu_2) + \mu_1^2 + \mu_2^2 + s^2 - 2\mu_2 s$$

Aside: completing squares

Completing squares

$$ax^2 + bx + c = a(x+m)^2 + n$$

$$m = \frac{b}{2a}$$

$$n = c - \frac{b^2}{4a}$$

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Completing squares

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$$q = 2x^2 - 2x(\mu_1 + s - \mu_2) + \mu_1^2 + \mu_2^2 + s^2 - 2\mu_2 s$$

$$= ax^2 + bx \qquad + c$$

$$q = 2\left(x - \frac{\mu_1 + s - \mu_2}{2}\right)^2$$

(left as an exercise)

$$f_S(s) = \frac{1}{\sqrt{4\pi\sigma^2}} e^{-\frac{\left(s - (\mu_1+\mu_2)\right)^2}{4\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma_s^2}} \cdot e^{-\frac{(s-\mu_s)^2}{2\sigma_s^2}}$$

$$= N(\mu_s, \sigma_s)$$

$$\mu_s = \mu_1 + \mu_2$$

$$\sigma_s^2 = 2\sigma^2$$