Transformation of Random Variables

ES 114

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Outline

What is a Random Variable?

Definition: A random variable (RV) is a function that maps outcomes in the sample space to real numbers.

Notation: $X : \Omega \to \mathbb{R}$

- Ω is the sample space (set of all possible outcomes).
- $X(\omega)$ assigns a number to each outcome ω.

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(This mapping lets us use tools like algebra, calculus, and probability theory on uncertain events.)

Discrete Random Variables

Example: Tossing a fair die

- Sample space: $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Define X: number shown on the die $\Rightarrow X : \Omega \to \{1, 2, 3, 4, 5, 6\}$
- PMF (Probability Mass Function): $p_X(k) = P(X = k)$

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For example:

$$P(X = 4) = \frac{1}{6}$$

(This is a pointwise probability: probability is assigned to each outcome. All probabilities add up to 1.)

Transformation Diagram

Sample Space
S with s_j

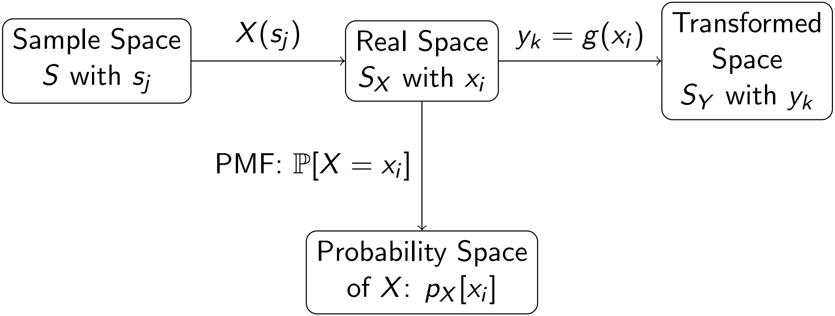
Transformation Diagram

$$\begin{array}{c}\text{Sample Space}\\ S \text{ with } s_j\end{array} \xrightarrow{\quad X(s_j) \quad} \begin{array}{c}\text{Real Space}\\ S_X \text{ with } x_i\end{array}$$

Transformation Diagram

$$\boxed{\begin{array}{c}\text{Sample Space}\\ S \text{ with } s_j\end{array}} \xrightarrow{X(s_j)} \boxed{\begin{array}{c}\text{Real Space}\\ S_X \text{ with } x_i\end{array}} \xrightarrow{y_k = g(x_i)} \boxed{\begin{array}{c}\text{Transformed}\\ \text{Space}\\ S_Y \text{ with } y_k\end{array}}$$

Transformation Diagram



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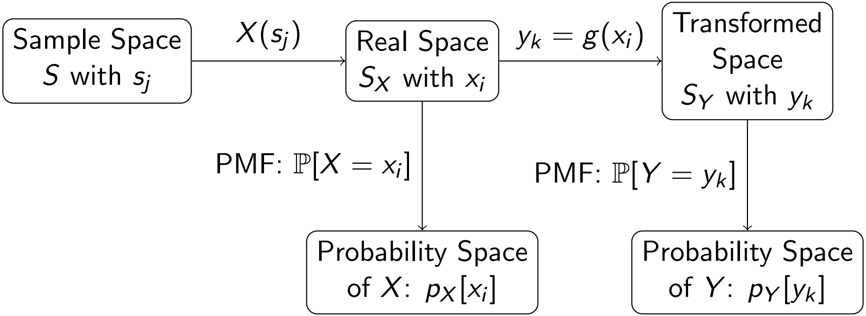

Example: Squaring a Discrete Random Variable

Given: A discrete random variable X:

X	$p_X(x)$
-2	$1/4$
-1	$1/4$
0	$1/8$
1	$1/8$
2	$1/4$

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Transformed Variable: $Y = X^2$:

Y	$p_Y(y)$
0	$1/8$
1	$3/8$
4	$4/8$

Die Toss Transformation

$$y_k = g(x_i) = \begin{cases} +1, & x_i \text{ even} \\ -1, & x_i \text{ odd} \end{cases}$$

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Real Space} \\ S_X = \{1, 2, 3, 4, 5, 6\}\end{array}} \arrow[r] \arrow[d, "\text{PMF}"'] & \boxed{\begin{array}{c}\text{Transformed Space} \\ S_Y = \{+1, -1\}\end{array}} \arrow[d, "\text{PMF}"'] \\
\boxed{p_X[x_i] = \frac{1}{6} \forall x_i} & \boxed{p_Y[y_k] = \frac{1}{2} \forall y_k}
\end{tikzcd}$$

Reminder 1: PDF (Probability Density Function)

- Denoted $f_X(x)$ for a random variable X.
- Tells you how "dense" the probability is near each point x.
- The area under $f_X(x)$ from $-\infty$ to some point a gives the probability of $X \leq a$.

Review: PDF vs. CDF

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Reminder 2: CDF (Cumulative Distribution Function)

- Denoted $F_X(x) = P(X \leq x)$.
- Always non-decreasing, starts at 0, goes up to 1.
- We often say $F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$.

(CDF is "Cumulative," PDF is "Point density." Think of PDF as the "speed," and CDF as the "distance traveled so far.")

Motivation: Transforming X into $Y = g(X)$

Given:

- A random variable X with known PDF $f_X(x)$ and CDF $F_X(x)$.
- A function $g(\cdot)$ (could be linear, quadratic, cosine, etc.).
- We define a new random variable: $Y = g(X)$.

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Wanted:

- The PDF of Y, $f_Y(y)$.
- The CDF of Y, $F_Y(y)$.

The CDF Trick: Step-by-Step

Step 1: Express $F_Y(y)$ in terms of X.

$$F_Y(y) = P\big(Y \leq y\big) = P\big(g(X) \leq y\big).$$

- Rewrite $g(X) \leq y$ in terms of conditions on X.
- Then you can use $F_X(x)$ or integrals of $f_X(x)$ to evaluate that probability.

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Step 2: Differentiate to find $f_Y(y)$.

$$f_Y(y) = \frac{d}{dy} F_Y(y).$$

Example 1: $Y = 2X + 3$

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Step 2: Find $f_Y(y)$.

$$f_Y(y) \;=\; \frac{d}{dy}F_Y(y) \;=\; \frac{1}{2}f_X\Big(\frac{y-3}{2}\Big).$$

Example 2: $Y = X^2$

Trickier, because squaring is <u>not</u> one-to-one:

$$Y = X^2 \quad \Longrightarrow \quad X = \pm\sqrt{Y}.$$

Step 1: $F_Y(y)$**.**

$$F_Y(y) = P\big(X^2 \le y\big) = P\big(-\sqrt{y} \;\le\; X \;\le\; \sqrt{y}\big) = F_X(\sqrt{y}) \;-\; F_X(-\sqrt{y}).$$

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Step 2: $f_Y(y)$**.**

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{2\sqrt{y}}\Big[f_X\big(\sqrt{y}\big) \,+\, f_X\big(-\sqrt{y}\big)\Big].$$

Example 2: Uniform X in $[a, b]$ (with $a > 0$)

If $X \sim$ **Uniform**$[a, b]$**, then** $Y = X^2 \in [a^2, b^2]$**.**

- CDF:

$$F_Y(y) = \frac{\sqrt{y} - a}{b - a}, \quad a^2 \le y \le b^2$$

  (careful with sign if $a > 0$).

- PDF:

$$f_Y(y) = \frac{1}{\sqrt{y}\,(b - a)}, \quad a^2 \le y \le b^2.$$

Example 3: $Y = \cos X$, with $X \sim \text{Uniform}[0, 2\pi]$

Step 1: $F_Y(y) = P(\cos X \leq y)$.

- This might feel like a geometry/trig puzzle.
- For $\cos X \leq y$, we find the ranges of X in $[0, 2\pi]$ that satisfy $\cos X \leq y$.

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Result (from standard trig):

$$F_Y(y) = 1 - \frac{\cos^{-1}(y)}{\pi}, \quad \text{for } -1 \leq y \leq 1.$$

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Step 2: Differentiate to find $f_Y(y)$.

$$f_Y(y) = \frac{1}{\pi\sqrt{1 - y^2}}, \quad -1 \leq y \leq 1.$$

Example 4: $Y = e^X$ with $f_X(x) = a\,e^x e^{-ae^x}$

Step 1: $F_Y(y)$**.**

$$F_Y(y) = P(Y \leq y) = P\big(e^X \leq y\big) = P\big(X \leq \ln y\big).$$

$$F_Y(y) = \int_{-\infty}^{\ln y} a\,e^x\,e^{-ae^x}\,dx.$$

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$$F_Y(y) = \int_{-\infty}^{\ln y} a\,e^x\,e^{-ae^x}\,dx.$$

Step 2: $f_Y(y)$.

- Use the substitution $u = e^x$. Then $x = \ln u$ and $dx = \frac{du}{u}$.
- After the dust settles, you'll get $f_Y(y) = a\,e^{-ay}$, an exponential(a) distribution.

("This wild PDF for X was cleverly chosen so that $Y = e^X$ becomes a plain old exponential distribution. Magic!")

Method Recap & Final Thoughts

General Recipe

1. Always start with CDF: $F_Y(y) = P\big(g(X) \le y\big)$.
2. Translate that event: rewrite in terms of X.
3. Use known PDFs or CDFs of X: for uniform, normal, or any distribution you have.
4. Differentiate to get $f_Y(y)$.

Big Hints

- If $g(\cdot)$ is invertible and monotonic, a direct formula (chain rule) is straightforward.
- If $g(\cdot)$ is not one-to-one (like squaring), break it into the relevant pieces (e.g. $+\sqrt{y}$ and $-\sqrt{y}$).
- *Draw pictures*: helps see "compression" or "expansion" of probability mass.

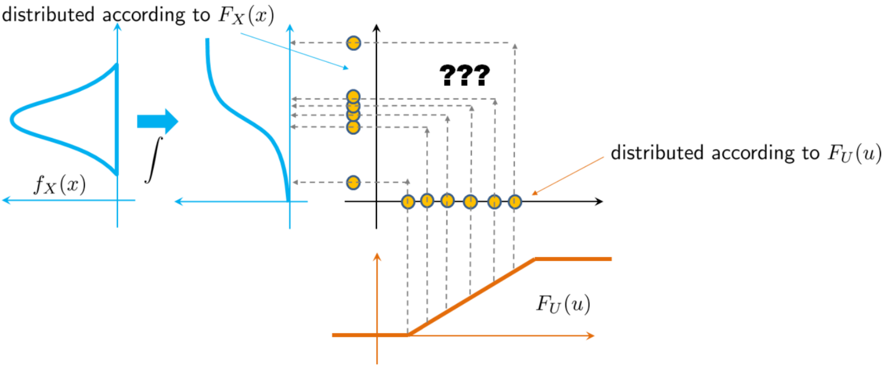
Problem Statement: Sampling from a Given Distribution

- Suppose we want to sample from a random variable X with a known cumulative distribution function (CDF) $F_X(x)$.
- Direct sampling may not be possible for arbitrary distributions.
- Can we use a simple random number generator (like uniform $U \sim \mathcal{U}(0,1)$) to generate samples of X?
- Yes! The **Inverse Transform Method** allows this by using the transformation:

$$X = F_X^{-1}(U).$$

Remember that $F_Y(y) = F_X(g^{-1}(y))$

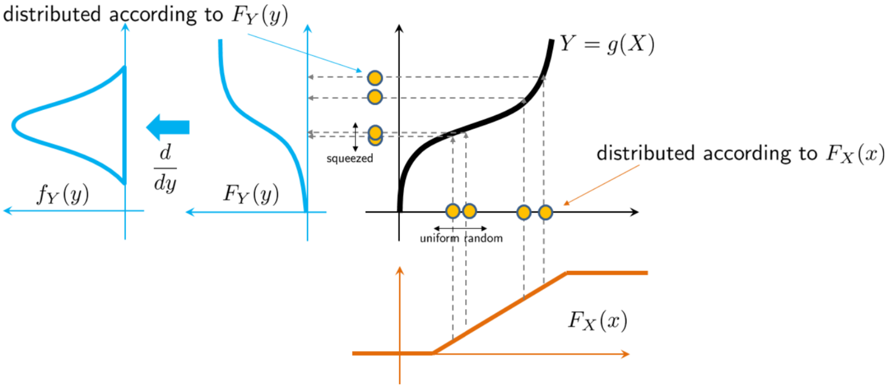
What are we looking for?



The inverse problem is: *By using what transformation g, such that $X = g(U)$, can we make sure that X is distributed according to $f_X(x)$ (or $F_X(x)$)?*

Looking at Y, the samples are traveling with g^{-1} in order to go back to F_X. Therefore, we need g^{-1} in the formula.

Why Does This Work?: Visualizing the Process



The transformation g that can turn a **uniform random variable** *into a random variable following a distribution $F_X(x)$ is given by*

$$g(u) = F^{-1}(u)$$

Use: $F_X(u) = u, \quad$ *for* $X \sim \mathcal{U}(0, 1)$

Q & A

Questions?

"Because the best way to make sense of a transformation is to ask what can go wrong!"